PRISMATIC DIEUDONNÉ THEORY

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Abstract. We define, for each quasi-syntomic ring R (in the sense of Bhatt-Morrow-Scholze), a category $\mathrm{DF}(R)$ of *filtered prismatic Dieudonné crystals over* R and a functor from p-divisible groups over R to $\mathrm{DF}(R)$. We prove that this functor is an antiequivalence when moreover R is flat over $\mathbb{Z}/p^n$ for some $n > 0$ or over $\mathbb{Z}_p$. Our main cohomological tool is the prismatic formalism recently developed by Bhatt and Scholze.

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1. Introduction

Let p be a prime number. The goal of the present paper is to establish classification theorems for p-divisible groups over *quasi-syntomic rings*. This class of rings is a non-Noetherian generalization of the class of p-complete locally complete intersection rings and contains also big rings, such as perfectoid rings. Our main theorem is as follows.

Theorem. *Let R be a quasi-syntomic ring. There is a natural functor from the category of p-divisible groups over R to the category $\mathrm{DF}(R)$ of filtered prismatic Dieudonné crystals over R. If moreover R is flat over $\mathbb{Z}/p^n$ (for some n) or $\mathbb{Z}_p$, this functor is an antiequivalence.*

A more precise version of this statement and a detailed explanation will be given later in this introduction. For now, let us just say that the category $\mathrm{DF}(R)$ is formed by objects of semi-linear algebraic nature.

The problem of classifying p-divisible groups and finite locally free group schemes by semi-linear algebraic structures has a long history, going back to the work of Dieudonné on formal groups over characteristic p perfect fields. In characteristic p, as envisionned by Grothendieck, and later developed by Messing ([42]), Mazur-Messing ([41]), Berthelot-Breen-Messing ([5], [6]), the formalism of crystalline cohomology provides a natural way to attach such invariants to p-divisible groups. This theory goes by the name of *crystalline Dieudonné theory* and leads to classification theorems for p-divisible groups over a characteristic p base in a wide variety of situations, which we will not try to survey but for which we refer the reader, for instance, to [37]. In mixed characteristic, the existing results have been more limited. Fontaine ([23]) obtained complete results when the base is the ring of integers of a finite totally ramified extension K of the ring of Witt vectors $W(k)$ of a perfect field k of characteristic p, with ramification index $e < p-1$. This ramification hypothesis was later removed by Breuil ([15]) for $p > 2$, who also conjectured an alternative reformulation of his classification in [14], simpler and likely to hold even for $p = 2$, which was proved by Kisin ([30], for odd p, and extended by Kim ([29]), Lau ([35]) and Liu ([38]) to all p. Zink gave a classification of *formal* p-divisible groups over very general bases using his theory of *displays* ([50]). More recently, p-divisible groups have been classified over perfectoid rings ([36], [47, Appendix to Lecture XVII]).

The main interest of our approach is that it gives a uniform and geometric construction of the classifying functor on quasi-syntomic rings. This is made possible by the recent spectacular work of Bhatt-Scholze on *prisms* and *prismatic cohomology* ([12], [7]). So far, such a cohomological construction of the functor had been available only in characteristic p, using the crystalline theory. This led in practice to some restrictions, when trying to study p-divisible groups in mixed characteristic by reduction to characteristic p, of which Breuil-Kisin theory is a prototypical example : there, no direct definition of the functor was available when $p = 2$! Replacing the crystalline formalism by the prismatic formalism, we give a definition of the classifying functor very close in spirit to the one used by Berthelot-Breen-Messing ([5]) and which now makes sense without the limitation to characteristic p. Over a quasi-syntomic ring R, our functor takes values in the category of *filtered prismatic Dieudonné crystals* over R (cf. Definition 4.1.6). As the name suggests, *prismatic Dieudonné crystals* are prismatic analogues of the classical notion of a

Dieudonné crystal on the crystalline site. The adjective *filtered* is here to indicate that one needs to add the datum of some kind of admissible filtration on the prismatic Dieudonné crystal.

Before stating precisely the main results of this paper and explaining the techniques involved, let us note that several natural questions are not addressed in this paper.

(1) It would be interesting to go beyond quasi-syntomic rings. By analogy with the characteristic p story, one would expect that the prismatic theory should also shed light on more general rings. In the general case, filtered prismatic Dieudonné crystals will not be the right objects to work with. One should instead define analogues of the *divided* Dieudonné crystals introduced recently by Lau [37] in characteristic p.
(2) Even for quasi-syntomic rings, our classification is explicit for the so called *quasi-regular semiperfectoid* rings or for complete regular local rings with perfect residue field of characteristic p (cf. Section 5.2), as will be explained below, but quite abstract in general. Classical Dieudonné crystals can be described as modules over the p-completion of the PD-envelope of a smooth presentation, together with a Frobenius and a connection satisfying various conditions. Is there an analogous concrete description of (filtered) prismatic Dieudonné crystals ?
(3) Finally, it would also be interesting and useful to study deformation theory (in the spirit of Grothendieck-Messing theory) for the filtered prismatic Dieudonné functor.

We now discuss in more detail the content of this paper.

1.1. **Quasi-syntomic rings.**

Let us first define the class of rings over which we study p-divisible groups.

Definition 1.1.1 (cf. Definition 3.3.1)**.** A ring R is *quasi-syntomic* if R is p-complete with bounded p^∞-torsion and if the cotangent complex $L_{R/\mathbb{Z}_p}$ has p-complete Tor-amplitude in $[-1,0]$[1]. The category of all quasi-syntomic rings is denoted by QSyn.

Similarly, a map $R \to R'$ of p-complete rings with bounded p^∞-torsion is a *quasi-syntomic morphism* if R' is p-completely flat over R and $L_{R'/R} \in D(R')$ has p-complete Tor-amplitude in $[-1,0]$.

Remark 1.1.2. This definition is due to Bhatt-Morrow-Scholze [11] and extends (in the p-complete world) the usual notion of l.c.i. rings and syntomic morphisms (flat and l.c.i.) to the non-Noetherian, non finite-type setting. The interest of this definition, apart from being more general, is that it more clearly shows why this category of rings is relevant : the key property of (quasi-)syntomic rings is that they have a well-behaved (p-completed) cotangent complex. The work of Avramov shows that the cotangent complex is very badly behaved for all other rings, at least in the Noetherian setting: it is left unbounded (cf. [2]).

Example 1.1.3. Any p-complete l.c.i. Noetherian ring is in QSyn. But there are also has big rings in QSyn : for example, any (integral) perfectoid ring is in QSyn (cf. Example 3.3.3). As a consequence of this, the p-completion of a smooth algebra

[1] This means that the complex $M = L_{R/\mathbb{Z}_p} \otimes^{\mathbb{L}}_R R/p \in D(R/p)$ is such that $M \otimes^{\mathbb{L}}_R N \in D^{[-1,0]}(R/p)$ for any R/p-module N.

over a perfectoid ring is also quasi-syntomic, as well as any bounded p^∞-torsion p-complete ring which can be presented as the quotient of an integral perfectoid ring by a finite regular sequence. For example, the rings

$$\mathcal{O}_C\langle T\rangle \quad ; \quad \mathcal{O}_{\mathbb{C}_p}/p \quad ; \quad \mathbb{F}_p[T^{1/p^\infty}]/(T-1)$$

are quasi-syntomic.

The category of quasi-syntomic rings is endowed with a natural topology : the Grothendieck topology for which covers are given by *quasi-syntomic covers*, i.e., morphisms $R \to R'$ of p-complete rings which are quasi-syntomic and p-completely faithfully flat.

An important property of the quasi-syntomic topology is that *quasi-regular semiperfectoid rings* form a basis of the topology (cf. Proposition 3.3.7).

Definition 1.1.4 (cf. Definition 3.3.5)**.** A ring R is *quasi-regular semiperfectoid* if $R \in \mathrm{QSyn}$ and there exists a perfectoid ring S mapping surjectively to R.

As an example, any perfectoid ring, or any p-complete bounded p^∞-torsion quotient of a perfectoid ring by a finite regular sequence, is quasi-regular semiperfectoid.

1.2. **Prisms and prismatic cohomology (after Bhatt-Scholze).** Our main tool for studying p-divisible groups over quasi-syntomic rings is the recent prismatic theory of Bhatt-Scholze [12], [7]. This theory relies on the seemingly simple notions of *δ-rings* and *prisms*. In what follows, all the rings considered are assumed to be $\mathbb{Z}_{(p)}$-algebras.

A *δ-ring* is a commutative ring A, together with a map of sets $\delta : A \to A$, with $\delta(0) = 0$, $\delta(1) = 0$, and satisfying the following identities :

$$\delta(xy) = x^p\delta(y)+y^p\delta(x)+p\delta(x)\delta(y) \quad ; \quad \delta(x+y) = \delta(x)+\delta(y)+\frac{x^p + y^p - (x+y)^p}{p},$$

for all $x, y \in A$. For any δ-ring (A, δ), denote by φ the map defined by

$$\varphi(x) = x^p + p\delta(x).$$

The identities satisfied by δ are made to make φ a ring endomorphism lifting Frobenius modulo p. Conversely, a p-torsion free ring equipped with a lift of Frobenius gives rise to a δ-ring. A pair (A, I) formed by a δ-ring A and an ideal $I \subset A$ is a *prism* if I defines a Cartier divisor on $\mathrm{Spec}(A)$, if A is (derived) (p, I)-complete and if I is pro-Zariski locally generated[2] by a distinguished element, i.e., an element d such that $\delta(d)$ is a unit.

Example 1.2.1.
(1) For any p-complete p-torsion free δ-ring A, the pair $(A, (p))$ is a prism.
(2) Say that a prism is *perfect* if the Frobenius φ on the undelying δ-ring is an isomorphism. Then the category of perfect prisms is equivalent to the category of (integral) perfectoid rings : in one direction, one maps a perfectoid ring R to the pair $(A_{\mathrm{inf}}(R) := W(R^\flat), \ker(\theta))$ (here $\theta : A_{\mathrm{inf}}(R) \to R$ is Fontaine's theta map) ; in the other direction, one maps (A, I) to A/I. Therefore, one sees that, in the words of the authors of [12], prisms are some kind of *"deperfection" of perfectoid rings*.

[2]In practice, the ideal I is always principal.

The crucial definition for us is the following. We stick to the affine case for simplicity, but it admits an immediate extension to p-adic formal schemes.

Definition 1.2.2. Let R be a p-complete ring. The *(absolute) prismatic site* $(R)_{\Delta}$ *of* R is the opposite of the category of bounded[3] prisms (A, I) together with a map $R \to A/I$, endowed with the Grothendieck topology for which covers are morphisms of prisms $(A, I) \to (B, J)$, such that the underlying ring map $A \to B$ is (p, I)-completely faithfully flat.

Bhatt and Scholze prove that the functor $\mathcal{O}_{\Delta}$ (resp. $\overline{\mathcal{O}}_{\Delta}$) on the prismatic site valued in (p, I)-complete δ-rings (resp. in p-complete R-algebras), sending $(A, I) \in (R)_{\Delta}$ to A (resp. A/I), is a sheaf. The sheaf $\mathcal{O}_{\Delta}$ (resp. $\overline{\mathcal{O}}_{\Delta}$) is called the *prismatic structure sheaf* (resp. the *reduced prismatic structure sheaf*).

From this, one easily deduces that the presheaves I_{Δ} (resp. $\mathcal{N}^{\geq 1}\mathcal{O}_{\Delta}$) sending (A, I) to I (resp. $\mathcal{N}^{\geq 1}A := \varphi^{-1}(I)$) are also sheaves on $(R)_{\Delta}$.

Let R be a p-complete ring. One proves the existence of a morphism of topoi :

$$v : \mathrm{Shv}((R)_{\Delta}) \to \mathrm{Shv}((R)_{\mathrm{qsyn}}).$$

Set :

$$\mathcal{O}^{\mathrm{pris}} := v_*\mathcal{O}_{\Delta} \ ; \ \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}} := v_*\mathcal{N}^{\geq 1}\mathcal{O}_{\Delta} \ ; \ \mathcal{I}^{\mathrm{pris}} := v_*I_{\Delta}.$$

The sheaf $\mathcal{O}^{\mathrm{pris}}$ is endowed with a Frobenius lift φ. Moreover, if R is quasi-syntomic, the quotient sheaf $\mathcal{O}^{\mathrm{pris}}/\mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}$ is naturally isomorphic to the structure sheaf $\mathcal{O}$ of $(R)_{\mathrm{qsyn}}$.

1.3. **Filtered prismatic Dieudonné crystals and modules.** We are now in position to define the category of objects classifying p-divisible groups.

Definition 1.3.1. Let R be a quasi-syntomic ring. A *filtered prismatic Dieudonné crystal over* R is a collection $(\mathcal{M}, \mathrm{Fil}\mathcal{M}, \varphi_{\mathcal{M}})$ consisting of a finite locally free $\mathcal{O}^{\mathrm{pris}}$-module $\mathcal{M}$, a $\mathcal{O}^{\mathrm{pris}}$-submodule $\mathrm{Fil}\mathcal{M}$, and a φ-linear map $\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$, satisfying the following conditions :

(1) $\varphi_{\mathcal{M}}(\mathrm{Fil}\mathcal{M}) \subset \mathcal{I}^{\mathrm{pris}}.\mathcal{M}$.
(2) $\mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}.\mathcal{M} \subset \mathrm{Fil}\mathcal{M}$ and $\mathcal{M}/\mathrm{Fil}\mathcal{M}$ is a finite locally free $\mathcal{O}$-module.
(3) $\varphi_{\mathcal{M}}(\mathrm{Fil}\mathcal{M})$ generates $\mathcal{I}^{\mathrm{pris}}.\mathcal{M}$ as an $\mathcal{O}^{\mathrm{pris}}$-module.

Definition 1.3.2. Let R be a quasi-syntomic ring. We denote by $\mathrm{DF}(R)$ the category of filtered prismatic Dieudonné crystals over R (with morphisms the $\mathcal{O}^{\mathrm{pris}}$-linear morphisms commuting with the Frobenius and respecting the filtration).

For quasi-regular semiperfectoid rings, these abstract objects have a concrete incarnation. Let R be a quasi-regular semiperfectoid ring. The prismatic site $(R)_{\Delta}$ admits a final object (Δ_R, I). Moreover, one has a natural isomorphism

$$\theta \colon \Delta_R/\mathcal{N}^{\geq 1}\Delta_R \cong R.$$

Example 1.3.3. (1) If R is a perfectoid ring, $(\Delta_R, I) = (A_{\mathrm{inf}}(R), \ker(\tilde{\theta}))$.
(2) If R is quasi-regular semiperfectoid and $pR = 0$, $(\Delta_R, I) \cong (A_{\mathrm{crys}}(R), (p))$.

[3]A prism (A, I) is *bounded* if A/I has bounded p^{∞}-torsion.

Definition 1.3.4. A *filtered prismatic Dieudonné module over R* is a collection $(M, \mathrm{Fil}\ M, \varphi_M)$ consisting of a finite locally free Δ_R-module M, a Δ_R-submodule $\mathrm{Fil}\ M$, and a φ-linear map $\varphi_M : M \to M$, satisfying the following conditions :

(1) $\varphi_M(\mathrm{Fil}\ M) \subset I.M$.
(2) $\mathcal{N}^{\geq 1}\Delta_R.M \subset \mathrm{Fil}\ M$ and $M/\mathrm{Fil}\ M$ is a finite locally free R-module.
(3) $\varphi_M(\mathrm{Fil}\ M)$ generates $I.M$ as a Δ_R-module.

Proposition 1.3.5 (Proposition 4.1.13). *Let R be a quasi-regular semiperfectoid ring. The functor of global sections induces an equivalence between the category of filtered prismatic Dieudonné crystals over R and the category of filtered prismatic Dieudonné modules over R.*

1.4. **Statements of the main results.** In all this paragraph, R is a quasi-syntomic ring.

Theorem 1.4.1 (Theorem 4.6.6). *Let G be a p-divisible group over R. The triple*

$$\left(\mathcal{M}_{\Delta}(G) = \mathcal{E}xt^1(G, \mathcal{O}^{\mathrm{pris}}), \mathrm{Fil}\mathcal{M}_{\Delta}(G) = \mathcal{E}xt^1(G, \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}), \varphi_{\mathcal{M}_{\Delta}(G)}\right)$$

where the $\mathcal{E}xt$ are Ext-groups of abelian sheaves on $(R)_{\mathrm{qsyn}}$ and $\varphi_{\mathcal{M}_{\Delta}(G)}$ is the Frobenius induced by the Frobenius of $\mathcal{O}^{\mathrm{pris}}$, is a filtered prismatic Dieudonné crystal over R, denoted by $\underline{\mathcal{M}}_{\Delta}(G)$.

Remark 1.4.2. When $pR = 0$, the *crystalline comparison theorem* for prismatic cohomology allows us to prove that this construction coincides with the functor usually considered in crystalline Dieudonné theory, relying on Berthelot-Breen-Messing's constructions ([5]).

Theorem 1.4.3 (Theorem 4.6.9). *Assume R is flat over $\mathbb{Z}/p^n$ (for some $n > 0$) or $\mathbb{Z}_p$. The filtered prismatic Dieudonné functor*

$$\underline{\mathcal{M}}_{\Delta} : G \mapsto \underline{\mathcal{M}}_{\Delta}(G)$$

induces an antiequivalence between the category $\mathrm{BT}(R)$ of p-divisible groups over R and the category $\mathrm{DF}(R)$ of filtered prismatic Dieudonné crystals over R.

Moreover, the prismatic Dieudonné functor :

$$\mathcal{M}_{\Delta} : G \mapsto \mathcal{M}_{\Delta}(G)$$

is fully faithful.

Remark 1.4.4. Theorem 1.4.1 (resp. Theorem 1.4.3) immediately extends to p-divisible groups over a quasi-syntomic formal scheme (resp. to p-divisible groups over a p-adic formal scheme quasi-syntomic over $\mathrm{Spf}(\mathbb{Z}/p^n)$, $n > 0$, or $\mathrm{Spf}(\mathbb{Z}_p)$).

Remark 1.4.5. It is easy to write down a formula for a functor attaching to a filtered prismatic Dieudonné crystal an abelian sheaf on $(R)_{\mathrm{qsyn}}$, which will be a quasi-inverse of the filtered prismatic Dieudonné functor : see Remark 4.9.6. But such a formula does not look very useful.

Remark 1.4.6. As a corollary of the theorem and the comparison with the crystalline functor, one obtains that the (contravariant) filtered Dieudonné functor from crystalline Dieudonné theory is an antiequivalence for quasi-syntomic rings in characteristic p. For excellent l.c.i. rings, fully faithfulness was proved by de Jong-Messing ; the antiequivalence was proved by Lau for F-finite l.c.i. rings (which are in particular excellent rings).

Remark 1.4.7. It is not difficult to prove that if R is perfectoid, filtered prismatic Dieudonné crystals (or modules) over R are equivalent to minuscule Breuil-Kisin-Fargues modules for R, in the sense of [10]. Moreover, to prove a classification theorem over arbitrary perfectoid rings, it is enough to do so for perfectoid rings which are either p-torsion free or of characteristic p, as explained in [36, Lemma 9.2, (9.2)]. Therefore, Theorem 1.4.3 contains as a special case the results of Lau and Scholze-Weinstein. But the proof of the theorem actually requires this special case[4] as an input.

Remark 1.4.8. In general, the prismatic Dieudonné functor (without the filtration) is not essentially surjective, but we prove it is an antiequivalence for p-complete regular (Noetherian) rings in Proposition 5.2.3.

Moreover, we explain in Section 5.2 how to recover Breuil-Kisin's classification (as extended by Kim, Lau and Liu to all p) of p-divisible groups over $\mathcal{O}_K$, where K is a discretely valued extension of $\mathbb{Q}_p$ with perfect residue field, from Theorem 1.4.3.

Remark 1.4.9. Section 5.3 shows how to extract from the filtered prismatic Dieudonné functor a functor from $\mathrm{BT}(R)$ to the category of displays of Zink over R. Even though the actual argument is slightly involved for technical reasons, the main result there ultimately comes from the following fact : if R is a quasi-regular semiperfectoid ring, the natural morphism $\theta : \Delta_R \to R$ gives rise by adjunction to a morphism of δ-rings $\Delta_R \to W(R)$, mapping $\mathcal{N}^{\geq 1}\Delta_R$ to the image of Verschiebung on Witt vectors.

Zink's classification by displays works on very general bases but is restricted (by design) to formal p-divisible groups or to odd p ; by contrast, our classification is limited to quasi-syntomic rings but do not make these restrictions.

Remark 1.4.10. As in Kisin's article [30], it should be possible to deduce from Theorem 1.4.3 a classification result for finite locally free group schemes. We only write this down over a perfectoid ring, in which case it was already known for $p > 2$ by the work of Lau, [36]. This result is used in forthwoming work of Česnavičius and Scholze [17].

1.5. **Overview of the proof and plan of the paper.** Section 2 and Section 3 contain some useful basic results concerning prisms and prismatic cohomology, with special emphasis on the case of quasi-syntomic rings. Most of them are extracted from [11] and [12], but some are not contained in loc. cit. (for instance, the definition of the q-logarithm, Section 2.2), or only briefly discussed there (for instance, the description of truncated Hodge-Tate cohomology, Section 3.2, or μ-torsion-freeness of prismatic cohomology of some quasi-regular semiperfectoid rings, Section 3.6).

Section 4 is the heart of this paper. We first introduce the category $\mathrm{DF}(R)$ of filtered prismatic Dieudonné crystals over a quasi-syntomic ring R and discuss some of its abstract properties (Section 4.1). We then introduce a candidate functor from p-divisible groups over R to $\mathrm{DF}(R)$ (Section 4.2). That it indeed takes values in the category $\mathrm{DF}(R)$ is the content of Theorem 1.4.1, which we do not prove immediately. We first relate this functor to other existing functors, for characteristic p rings or perfectoid rings (Section 4.3). The next three sections are devoted to the proof of Theorem 1.4.1. This proof follows a road similar to the one of [5, Ch. 2,

[4] In fact, as observed in [47], only the case of perfectoid valuation rings with algebraically closed and spherically complete fraction field is needed.

3]. The basic idea is to reduce many statements to the case of p-divisible groups attached to abelian schemes, using a theorem of Raynaud ensuring that a finite locally free group scheme on R can always be realized as the kernel of an isogeny between two abelian schemes over R, Zariski-locally on R. For abelian schemes, via the general device, explained in [5, Ch. 2] and recalled in Section 4.4, for computing Ext-groups in low degrees in a topos, one needs a good understanding of the prismatic cohomology. It relies on the degeneration of the conjugate spectral sequence abutting to reduced prismatic cohomology, in the same way as the description of the crystalline cohomology of abelian schemes is based on the degeneration of the Hodge-de Rham spectral sequence. We prove it in Section 4.5 by appealing to the identification of some truncation of the reduced prismatic complex with some cotangent complex, in the spirit of Deligne-Illusie (or, more recently, [10]), proved in Section 3.2. The prismatic perspective provides a very natural way of doing this.

To prove Theorem 1.4.3, stated as Theorem 4.6.9 below, one first observes that the functors

$$R \mapsto \mathrm{BT}(R) \quad ; \quad R \mapsto \mathrm{DF}(R)$$

on QSyn are both stacks for the quasi-syntomic topology (for BT, this is done in the Appendix). Therefore, to prove that the functor $\underline{\mathcal{M}}_{\Delta}$ is an antiequivalence, it is enough to prove it for R quasi-regular semiperfectoid, since these rings form a basis of the topology, in which case one can simply consider the more concrete functor $\underline{M}_{\Delta}$ taking values in filtered prismatic Dieudonné modules over R, defined by taking global sections of $\underline{\mathcal{M}}_{\Delta}$. Therefore, one sees that, even if one is ultimately interested only by Noetherian rings, the structure of the argument forces to consider large quasi-syntomic rings[5].

Assume from now on that R is quasi-regular semiperfectoid. Fully faithfulness is shown using the strategy of [48] (following an idea of de Jong-Messing) : one first proves fully faithfulness for morphisms from $\mathbb{Q}_p/\mathbb{Z}_p$ to μ_{p^∞} and then reduces to this special case. The first step is actually delicate and unfortunately relies on some results on algebraic K-theory from [18] and the companion paper [1], due to the fact that general quasi-regular semiperfectoid rings are quite hard to handle directly, contrary to the ones which are obtained as colimits of quotients of perfectoid rings by finite regular sequences. We were not able to obtain a more direct proof. Nevertheless, the proof works in general. By contrast, the second step requires the additional hypothesis of Theorem 1.4.3 on R. It is inspired by the analogous step in Scholze-Weinstein's paper ([48, §4.3]), but more direct and less intricate, even when the ring R is an $\mathbb{F}_p$-algebra. In fact, we even prove under theses assumptions the (a priori, but not a posteriori) stronger result that the prismatic Dieudonné functor (forgetting the filtration) is fully faithful.

Once fully faithfulness is acquired, the proof of essential surjectivity is by reduction to the perfectoid case. One can actually even reduce to the case of perfectoid valuation rings with algebraically closed fraction field. In this case, the result is known, and due - depending whether one is in characteristic p or in mixed characteristic - to Berthelot and Scholze-Weinstein.

Finally, Section 5 gathers several complements to the main theorems, already mentioned above : the classification of finite locally free group schemes of p-power order over a perfectoid ring, Breuil-Kisin's classification of p-divisible groups over

[5]In characteristic p, Lau has recently and independently implemented a similar strategy in [37].

the ring of integers of a finite extension of $\mathbb{Q}_p$, the relation with the theory of displays and the description of the Tate module of the generic fiber of a p-divisible group from its prismatic Dieudonné crystal.

1.6. **Notations and conventions.** In all the text, we fix a prime number p.

- All finite locally free group schemes will be assumed to be commutative.
- If R is a ring, we denote by $\mathrm{BT}(R)$ the category of p-divisible groups over R.
- If A is a ring, $I \subset A$ an ideal, and $K \in D(A)$ an object of the derived category of A-modules, K is said to be *derived I-complete* if for every $f \in I$, the derived limit of the inverse system
$$\dots K \xrightarrow{f} K \xrightarrow{f} K$$
vanishes. Equivalently, when $I = (f_1, \dots, f_r)$ is finitely generated, K is derived I-complete if the natural map
$$K \to R\lim(K \otimes_A^{\mathbb{L}} K_n^\bullet)$$
is an isomorphism in $D(A)$, where for each $n \geq 1$, $K_n^\bullet$ denotes the Koszul complex $K_\bullet(A; f_1^n, \dots, f_r^n)$ (one has $H^0(K_n^\bullet) = A/(f_1^n, \dots, f_r^n)$, but beware that in general $K_n^\bullet$ may also have cohomology in negative degrees, unless $(f_1, \dots, f_r)$ forms a regular sequence). An A-module M is said to be *derived I-complete* if $K = M[0] \in D(A)$ is derived I-complete. The following properties are useful in practice :
  (1) A complex $K \in D(A)$ is derived I-complete if and only if for each integer i, $H^i(K)$ is derived I-complete (this implies in particular that the category of derived I-complete A-modules form a weak Serre subcategory of the category of A-modules).
  (2) If $I = (f_1, \dots, f_r)$ is finitely generated, the inclusion of the full subcategory of derived I-complete complexes in $D(A)$ admits a left adjoint, sending $K \in D(A)$ to its *derived I-comletion*
$$\widehat{K} = R\lim(K \otimes_A^{\mathbb{L}} K_n^\bullet).$$
  Note that the derived I-completion of an A-module M (viewed as a complex sitting in degree 0) need not be discrete.
  (3) (Derived Nakayama) If I is finitely generated, a derived I-complete complex $K \in D(A)$ (resp. a derived I-complete A-module M) is zero if and only if $K \otimes_A^{\mathbb{L}} A/I = 0$ (resp. $M/IM = 0$).
  (4) If I is finitely generated, an A-module M is (classically) I-adically complete if and only if it is derived I-complete and I-adically separated.
  (5) $I = (f)$ is principal and M is an A-module with bounded f^∞-torsion (i.e. such that $M[f^\infty] = M[f^N]$ for some N), the derived I-completion of M (as a complex) is discrete and coincides with its (classical) I-adic completion.

  A useful reference for derived completions is [49, Tag 091N].
- Let A be a ring, I a finitely generated ideal. A complex $K \in D(A)$ is *I-completely flat* (resp. *I-completely faithfully flat*) if $K \otimes_A^{\mathbb{L}} A/I$ is concentrated in degree 0 and flat (resp. faithfully flat), cf. [11, Definition 4.1.]. If an A-module M is flat, its derived completion $\widehat{M}$ is I-completely flat.

Assume that I is principal, generated by $f \in A$ (in the sequel, f will often be p). Let $A \to B$ be a map of derived f-complete rings. If A has bounded f^∞-torsion and $A \to B$ is f-completely flat, then B has bounded f^∞-torsion. Conversely, if B has bounded f^∞-torsion and $A \to B$ is f-completely faithfully flat, A has bounded f^∞-torsion. Moreover, if A and B both have bounded f^∞-torsion, then $A \to B$ is f-completely (faithfully) flat if and only if $A/f^n \to B/f^n$ is (faithfully) flat for all $n \geq 1$. See [11, Corollary 4.8]).

- A derived I-complete A-algebra R is *I-completely étale* (resp. *I-completely smooth*) if $R \otimes_A^{\mathbb{L}} A/I$ is concentrated in degree 0 and étale (resp. smooth).

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2. Generalities on prisms

In this section we review the theory of prisms and collect some additional results. In particular, we present the definition of the q-logarithm (cf. Section 2.2).

2.1. Prisms and perfectoid rings.

We list here some basic definitions and results from [12], of which we will make constant use in the paper. Let us first recall the definition of a δ-ring A. In the following all rings will be assumed to be $\mathbb{Z}_{(p)}$-algebras.

Definition 2.1.1. A *δ-ring* is a pair (A,δ) with A a commutative ring and $\delta\colon A\to A$ a map (of sets) such that for $x,y\in A$ the following equalities hold:

$$\delta(0)=\delta(1)=0$$
$$\delta(xy)=x^p\delta(y)+y^p\delta(x)+p\delta(x)\delta(y)$$
$$\delta(x+y)=\delta(x)+\delta(y)+\tfrac{x^p+y^p-(x+y)^p}{p}.$$

A morphism of δ-rings $f\colon(A,\delta)\to(A',\delta')$ is a morphism $f\colon A\to A'$ of rings such that $f\circ\delta=\delta'\circ f$.

By design the morphism

$$\varphi\colon A\to A,\ x\mapsto x^p+p\delta(x)$$

for a δ-ring (A,δ) is a ring homomorphism lifting the Frobenius on A/p. Using φ the second property of δ can be rephrased as

$$\delta(xy)=\varphi(x)\delta(y)+y^p\delta(x)=x^p\delta(y)+\varphi(y)\delta(x)$$

which looks close to that of a derivation. If A is p-torsion free, then any Frobenius lift $\psi\colon A\to A$ defines a δ-structure on A by setting

$$\delta(x):=\frac{\psi(x)-x^p}{p}.$$

Thus, in the p-torsion free case a δ-ring is the same as a ring with a Frobenius lift.

Remark 2.1.2. The category of δ-rings has all limits and colimits and that these are calculated on the underlying rings[6] (cf. [12, Section 1]). In particular, there exists free δ-rings (by the adjoint functor theorem). Concretely, if A is a δ-ring and X is a set, then the free δ-ring $A\{X\}$ on X is a polynomial ring over A with variables $\delta^n(x)$ for $n\geq 0$ and $x\in X$ (cf. [12, Lemma 2.11]). Moreover, the Frobenius on $\mathbb{Z}_{(p)}\{X\}$ is faithfully flat (cf. [12, Lemma 2.11]).

Definition 2.1.3. Let (A,δ) be a δ-ring.

(1) An element $x\in A$ is called *of rank* 1 if $\delta(x)=0$.
(2) An element $d\in A$ is called *distinguished* if $\delta(d)\in A^\times$ is a unit.

In particular, $\varphi(x)=x^p$ if $x\in A$ is of rank 1.

Here is a useful lemma showing how to find rank 1 elements in a p-adically separated δ-ring.

Lemma 2.1.4. *Let A be a δ-ring and let $x\in A$. Then $\delta(x^{p^n})\in p^nA$ for all n. In particular, if A is p-adically separated and $y\in A$ admits a p^n-th root for all $n\geq 0$, then $\delta(y)=0$, i.e., y has rank* 1.

Proof. Cf. [12, Lemma 2.31]. □

[6] This does not hold for the category of rings with a Frobenius lift in the presence of p-torsion.

We can now state the definition of a prism (cf. [12, Definition 3.2]). Recall that a δ-pair (A, I) is simply a δ-ring A together with an ideal $I \subseteq A$.

Definition 2.1.5. A δ-pair (A, I) is a *prism* if $I \subseteq A$ is an invertible ideal such that A is derived (p, I)-complete, and $p \in I + \varphi(I)A$. A prism (A, I) is called *bounded* if A/I has bounded p^∞-torsion.

Remark 2.1.6. Some comments about these definitions are in order :

(1) By [12, Lemma 3.1] the condition $p \in I + \varphi(I)A$ is equivalent to the fact that I is pro-Zariski locally on $\mathrm{Spec}(A)$ generated by a distinguished element. Thus it is usually not much harm to assume that $I = (d)$ is actually principal[7].

(2) If $(A, I) \to (B, J)$ is a morphism of prisms, i.e., $A \to B$ is a morphism of δ-rings carrying I to J, then [12, Lemma 3.5] implies that $J = IB$.

(3) An important example of a prism is provided by
$$(A, I) = (\mathbb{Z}_p[[q-1]], ([p]_q))$$
where
$$[p]_q := \frac{q^p - 1}{q - 1}$$
is the q-analog of p. Many other interesting examples will appear below.

(4) The prism (A, I) being bounded implies that A is classically (p, I)-adically complete (cf. [7, Exercise 3.4.]), and thus in particular p-adically separated.

Lemma 2.1.7. *Let (A, I) be a prism and let $d \in I$ be distinguished. If (p, d) is a regular sequence in A, then for all $r, s \geq 0$, $r \neq s$ the sequences*
$$(p, \varphi^r(d)), (\varphi^r(d), \varphi^s(d))$$
are regular.

Proof. Note that it suffices to consider the case $s = 0$ by assuming $r > s$ and replacing d by $\varphi^s(d)$. Then the statement is proven in [1, Lemma 3.3] and [1, Lemma 3.6]. $\square$

Previous work in p-adic Hodge theory used, in one form or another, the theory of perfectoid spaces. From the prismatic perspective, this is explained as follows. We recall that a δ-ring A (or prism (A, I)) is called *perfect* if the Frobenius $\varphi\colon A \to A$ is an isomorphism. If A is perfect, then necessarily $A \cong W(R)$ for some perfect ring R (cf. [12, Corollary 2.30]).

Proposition 2.1.8. *The functor*
$$\{\textit{perfect prisms } (A, I)\} \to \{\textit{(integral) perfectoid rings } R\},\ (A, I) \mapsto A/I.$$
is an equivalence of categories with inverse $R \mapsto (A_{\mathrm{inf}}(R), \ker(\tilde{\theta}))$, where $A_{\mathrm{inf}}(R) := W(R^\flat)$ and $\tilde{\theta} = \theta \circ \varphi^{-1}$, θ being Fontaine's theta map.

Proof. Cf. [12, Theorem 3.9]. $\square$

Remark 2.1.9. (1) Of course, one could use θ instead of $\tilde{\theta}$. We make this (slightly strange) choice for coherence with later choices.

(2) The theorem implies in particular that for every perfect prism (A, I), the ideal I is principal.

[7]For example, if A is perfect, i.e., the Frobenius $\varphi\colon A \to A$ is bijective, then this condition is automatic by [12, Lemma 3.7].

As a corollary, we get the following easy case of almost purity.

Corollary 2.1.10. *Let R be a perfectoid ring and let $R \to R'$ be p-completely étale. Then R' is perfectoid. Moreover, if $J \subseteq R$ is an ideal, then the p-completion R' of the henselization of R at J is perfectoid.*

Proof. We can lift R' to a $(p, \ker(\theta))$-completely étale $A_{\inf}(R)$-algebra B. By [12, Lemma 2.18], the δ-structure on $A_{\inf}(R)$ extends uniquely to B. Reducing modulo p we see that B is a perfect δ-ring as it is $(p, \ker(\theta))$-completely étale over $A_{\inf}(R)$. Using Proposition 2.1.8 $R' \cong B/\ker(\theta)B$ is therefore perfectoid. The statement on henselizations follows from this as henselizations are colimits along étale maps (cf. the proof of [49, Tag 0A02]). □

Moreover, perfectoid rings enjoy the following fundamental property.

Proposition 2.1.11. *Let (A, I) be a perfect prism. Then for every prism (B, J) the map*

$$\mathrm{Hom}((A, I), (B, J)) \to \mathrm{Hom}(A/I, B/J)$$

is a bijection.

Proof. Cf. [12, Lemma 4.7.]. □

2.2. **The q-logarithm.** Each prism is endowed with its *Nygaard filtration* (cf. [7, Definition 11.2.]).

Definition 2.2.1. Let (A, I) be a prism. Then we set

$$\mathcal{N}^{\geq i}A := \varphi^{-1}(I^i)$$

for $i \geq 0$. The filtration $\mathcal{N}^{\geq \bullet}A$ is called the *Nygaard filtration of* (A, I).

This filtration (or rather the first piece of this filtration) will play an important role in the rest of this text. It already shows up when proving the existence of the q-logarithm

$$\log_q \colon \mathbb{Z}_p(1)(B/J) \to B,\ x \mapsto \log_q([x^{1/p}]_{\tilde{\theta}})$$

for a prism (A, I) over $(\mathbb{Z}_p[[q-1]], ([p]_q))$ from Remark 2.1.6, as we now explain.

Here,

$$\mathbb{Z}_p(1) := T_p(\mu_{p^\infty})$$

is the functor sending a ring R to $T_p(R^\times) = \varprojlim_n \mu_{p^n}(R)$ and

$$[-]_{\tilde{\theta}} \colon \varprojlim_{x \mapsto x^p} A/I \to A$$

is the Teichmüller lift sending a p-power compatible system

$$x := (x_0, x_1, \ldots) \in \varprojlim_{x \mapsto x^p} A/I$$

to the limit

$$[x]_{\tilde{\theta}} := \varinjlim_{n \to \infty} \tilde{x}_n^{p^n}$$

where $\tilde{x}_n \in A$ is a lift of $x_n \in A/I$. By definition,

$$\mathbb{Z}_p(1)(A/I) \subseteq \varprojlim_{x \mapsto x^p} A/I, \quad (x_0, x_1, \ldots) \mapsto (1, x_0, x_1, \ldots).$$

Moreover, on $\varprojlim_{x\mapsto x^p} A/I$ one can take p-th roots

$$(-)^{1/p}\colon \varprojlim_{x\mapsto x^p} A/I \to \varprojlim_{x\mapsto x^p} A/I,\ (x_0,x_1,\ldots)\mapsto (x_1,x_2,\ldots).$$

In [1, Lemma 4.10] there is the following lemma on the q-logarithm. For $n\in\mathbb{Z}$ we recall that the q-number $[n]_q$ is defined as

$$[n]_q := \frac{q^n-1}{q-1}\in \mathbb{Z}_p[[q-1]].$$

Lemma 2.2.2. *Let (B,J) be a prism over $(\mathbb{Z}_p[[q-1]],([p]_q))$. Then for every element $x\in 1+\mathcal{N}^{\geq 1}B$ of rank 1, i.e., $\delta(x)=0$, the series*

$$\log_q(x)=\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(n-1)/2}\frac{(x-1)(x-q)\cdots(x-q^{n-1})}{[n]_q}$$

is well-defined and converges in B. Moreover, $\log_q(x)\in\mathcal{N}^{\geq 1}B$ and $\log_q(x)=\frac{q-1}{\log(q)}\log(x)$, where $\log(x):=\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)}{n}$[8].

The defining properties of the q-logarithm are that $\log_q(1)=0$ and that its q-derivative is $\frac{d_q x}{x}$ (cf. [1, Lemma 4.6.]).

One derives easily the existence of the "divided q-logarithm".

Lemma 2.2.3. *Let (B,J) be a prism over $(\mathbb{Z}_p[[q-1]],([p]_q))$ and let $x\in\mathbb{Z}_p(1)(B/J)$. Then $[x^{1/p}]_{\tilde\theta}\in B$ is of rank 1 and lies in $1+\mathcal{N}^{\geq 1}B$. Thus*

$$\log_q([x^{1/p}]_{\tilde\theta})=\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(n-1)/2}\frac{([x^{1/p}]_{\tilde\theta}-1)\ldots([x^{1/p}]_{\tilde\theta}-q^{n-1})}{[n]_q}$$

exists in B.

Proof. By Lemma 2.1.4 the element $[x^{1/p}]_{\tilde\theta}$ is of rank 1 as it admits arbitrary p^n-roots. Moreover, $[x^{1/p}]_{\tilde\theta}\in 1+\mathcal{N}^{\geq 1}B$ as $\varphi([x^{1/p}]_{\tilde\theta})=[x]_{\tilde\theta}\equiv 1$ modulo J. By Lemma 2.2.2 we can therefore conclude. ∎

[8] This equality holds in $B[1/p][[x-1]]^{\wedge(q-1)}$.

3. Generalities on prismatic cohomology

3.1. **Prismatic site and prismatic cohomology.** In this paragraph, we shortly recall, mostly for the convenience of the reader and to fix notations, some fundamental definitions and results, without proofs, from [12]. Fix a bounded prism (A, I). Let R be a p-complete A/I-algebra.

Definition 3.1.1. The *prismatic site of R relative to A*, denoted $(R/A)_{\mathbb{\Delta}}$, is the category whose objects are given by bounded prisms (B, IB) over (A, I) together with an A/I-algebra map $R \to B/IB$, with the obvious morphisms, endowed with the Grothendieck topology for which covers are given by (p, I)-completely faithfully flat morphisms of prisms over (A, I).

Remark 3.1.2. In this remark we deal with the set-theoretic issues arising from Definition 3.1.1. For example, as it stands there does not exist a sheafification functor for presheaves on $(R/A)_{\mathbb{\Delta}}$. We will implicitly fix a cut-off cardinal κ like in [46, Lemma 4.1] and assume that all objects appearing in Definition 3.1.1 (or Definition 3.1.4) have cardinality $< \kappa$. The results of this paper will not change under enlarging κ. For example, the category of prismatic Dieudonné crystals on $(R)_{\mathbb{\Delta}}$ will be independent of the choice of κ. Also the prismatic cohomology does not change (because it can be calculated via a Čech-Alexander complex), and thus the (filtered) prismatic Dieudonné crystals will be independent of κ (by Section 4.4).

This affine definition admits an immediate extension to p-adic formal schemes over $\mathrm{Spf}(A/I)$, cf [12].

Proposition 3.1.3 ([12], Corollary 3.12)**.** *The functor $\mathcal{O}_{\mathbb{\Delta}}$ (resp. $\overline{\mathcal{O}}_{\mathbb{\Delta}}$) on the prismatic site valued in (p, I)-complete $\delta - A$-algebras (resp. in p-complete R-algebras), sending $(B, IB) \in (R/A)_{\mathbb{\Delta}}$ to B (resp. B/IB), is a sheaf. The sheaf $\mathcal{O}_{\mathbb{\Delta}}$ (resp. $\overline{\mathcal{O}}_{\mathbb{\Delta}}$) is called the prismatic structure sheaf (resp. the reduced prismatic structure sheaf).*

These constructions have absolute variants, where one does not fix a base prism. Let R be a p-complete ring.

Definition 3.1.4. The *(absolute) prismatic site of* R, denoted $(R)_{\mathbb{\Delta}}$, is the category whose objects are given by bounded prisms (B, J) together with a ring map $R \to B/J$, with the obvious morphisms, endowed with the Grothendieck topology for which covers are given by morphisms of prism $(B, J) \to (C, JC)$ which are (p, I)-completely faithfully flat.

Exactly as before, one defines functors $\mathcal{O}_{\mathbb{\Delta}}$ and $\overline{\mathcal{O}}_{\mathbb{\Delta}}$, which are sheaves on $(R)_{\mathbb{\Delta}}$.

We turn to the definition of (derived) prismatic cohomology. Fix a bounded prism (A, I). The prismatic cohomology of R over A is defined in two steps. One starts with the case where R is p-completely smooth over A/I.

Definition 3.1.5. Let R be a p-complete p-completely smooth A/I-algebra. The prismatic complex $\mathbb{\Delta}_{R/A}$ of R over A is defined to be the cohomology of the sheaf $\mathcal{O}_{\mathbb{\Delta}}$ on the prismatic site :

$$\mathbb{\Delta}_{R/A} = R\Gamma((R/A)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}).$$

This is a (p, I)-complete commutative algebra object in $D(A)$ endowed with a semilinear map $\varphi : \mathbb{\Delta}_{R/A} \to \mathbb{\Delta}_{R/A}$, induced by the Frobenius of $\mathcal{O}_{\mathbb{\Delta}}$.

Similarly, one defines the reduced prismatic complex or Hodge-Tate complex :

$$\overline{\Delta}_{R/A} = R\Gamma((R/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta}).$$

This is a p-complete commutative algebra object in $D(R)$.

A fundamental property of prismatic cohomology is the Hodge-Tate comparison theorem, which relates the Hodge-Tate complex to differential forms. For this, first recall that for any A/I-module M and integer n, the nth-Breuil-Kisin twist of M is defined as

$$M\{n\} := M \otimes_{A/I} (I/I^2)^{\otimes n}.$$

The Bockstein maps

$$\beta_I : H^i(\overline{\Delta}_{R/A})\{i\} \to H^{i+1}(\overline{\Delta}_{R/A})\{i+1\}$$

for each $i \geq 0$ make $(H^*(\overline{\Delta}_{R/A}\{*\}, \beta_I)$ a graded commutative A/I-differential graded algebra[9], which comes with a map $\eta \colon R \to H^0(\overline{\Delta}_{R/A})$.

Theorem 3.1.6 ([12], Theorem 4.10)**.** *The map η extends to a map*

$$\eta_R^* : (\Omega^{\wedge_p}_{R/(A/I)}, d) \to (H^*(\overline{\Delta}_{R/A}, \beta_I)$$

which is an isomorphism.

While proving Theorem 3.1.6, Bhatt and Scholze also relate prismatic and crystalline cohomology when the ring R is an $\mathbb{F}_p$-algebra. The precise statement is the following. Assume that $I = (p)$, i.e. that (A, I) is a crystalline prism. Let $J \subset A$ be a PD-ideal with $p \in J$. Let R be a smooth A/J-algebra and

$$R^{(1)} = R \otimes_{A/J} A/p,$$

where the map $A/J \to A/p$ is the map induced by Frobenius and the fact that J is a PD-ideal.

Theorem 3.1.7 ([12], Theorem 5.2)**.** *Under the previous assumptions, there is a canonical isomorphism of $E_\infty - A$-algebras*

$$\Delta_{R^{(1)}/A} \simeq R\Gamma_{\mathrm{crys}}(R/A),$$

compatible with Frobenius.

Remark 3.1.8. (1) If $J = (p)$, $R^{(1)}$ is just the Frobenius twist of R.
(2) The proof of Theorem 3.1.7 goes through for a syntomic A/J-algebra R. The important point is that in the proof in [12, Theorem 5.2] in each simplicial degree the kernel of the morphism $B^\bullet \to \tilde{R}$ is the inductive limit of ideals of the form $(p, x_1, \ldots, x_r)$ with $(x_1, \ldots, x_r)$ being p-completely regular relative to A, which allows to apply [12, Proposition 3.13]. The statement extends by descent from the quasi-regular semiperfect case to all quasi-syntomic rings over $\mathbb{F}_p$ (cf. Lemma 3.4.3).

Definition 3.1.5 of course makes sense without the hypothesis that R is p-completely smooth over A/I. But it would not give well behaved objects ; for instance, the Hodge-Tate comparison would not hold in general[10].. The formalism

[9]For $p = 2$ this assertion is non-trivial and part of the proof of [12, Theorem 4.10].

[10]Nevertheless, in Section 3.4 we will check that the site-theoretic defined prismatic cohomology is well-behaved for quasi-regular semiperfectoid rings (as it agrees with the derived prismatic cohomology), and also for quasi-syntomic rings

of non-abelian derived functors allows to extend the definition of the prismatic and Hodge-Tate complexes to all p-complete A/I-algebras in a manner compatible with the Hodge-Tate comparison theorem.

Definition 3.1.9. The *derived prismatic cohomology* functor $L\Delta_{-/A}$ (resp. the *derived Hodge-Tate cohomology* functor $L\overline{\Delta}_{-/A}$) is the left Kan extension (cf. [11, Construction 2.1]) of the functor $\Delta_{-/A}$ (resp. $\overline{\Delta}_{-/A}$) from p-completely smooth A/I-algebras to (p, I)-complete commutative algebra objects in (the ∞-category) $D(A)$ (resp. p-complete commutative algebra objects in $D(R)$), to the category of p-complete A/I-algebras.

For short, we will just write $\Delta_{R/A}$ (resp. $\overline{\Delta}_{R/A}$) for $L\Delta_{R/A}$ (resp. $L\overline{\Delta}_{R/A}$) in the following.

Left Kan extension of the Postnikov (or canonical filtration) filtration leads to an extension of Hodge-Tate comparison to derived prismatic cohomology.

Proposition 3.1.10. *For any p-complete A/I-algebra R, the derived Hodge-Tate complex $\overline{\Delta}_{R/A}$ comes equipped with a functorial increasing multiplicative exhaustive filtration $\mathrm{Fil}_*^{\mathrm{conj}}$ in the category of p-complete objects in $D(R)$ and canonical identifications*

$$\mathrm{gr}_i^{\mathrm{conj}}(\overline{\Delta}_{R/A}) \simeq \wedge^i L_{R/(A/I)}\{-i\}[-i]^{\wedge_p}.$$

Finally, let us indicate how these affine statements globalize.

Proposition 3.1.11. *Let X be a p-adic formal scheme over $\mathrm{Spf}(A/I)$. There exists a functorially defined (p, I)-complete commutative algebra object $\Delta_{X/A} \in D(X, A)$, equipped with a φ_A-linear map $\varphi_X : \Delta_{X/A} \to \Delta_{X/A}$, and having the following properties :*

- *For any affine open $U = \mathrm{Spf}(R)$ in X, there is a natural isomorphism of (p, I)-complete commutative algebra objects in $D(A)$ between $R\Gamma(U, \Delta_{X/A})$ and $\Delta_{R/A}$, compatible with Frobenius.*
- *Set $\overline{\Delta}_{X/A} = \Delta_{X/A} \otimes^{\mathbb{L}}_A A/I \in D(X, A/I)$. Then $\overline{\Delta}_{X/A}$ is naturally an object of $D(X)$, which comes with a functorial increasing multiplicative exhaustive filtration $\mathrm{Fil}_*^{\mathrm{conj}}$ in the category of p-complete objects in $D(X)$ and canonical identifications*

$$\mathrm{gr}_i^{\mathrm{conj}}(\overline{\Delta}_{X/A}) \simeq \wedge^i L_{X/(A/I)}\{-i\}[-i]^{\wedge_p}.$$

3.2. **Truncated Hodge-Tate cohomology and the cotangent complex.** Let (A, I) be a bounded prism, and let X be a p-adic A/I-formal scheme. The following result also appears in [12, Proposition 4.14]. We give a similar argument (suggested to us by Bhatt), with more details than in loc. cit.

Proposition 3.2.1. *There is a canonical isomorphism :*

$$\alpha_X : L_{X/\mathrm{Spf}(A)}\{-1\}[-1]^{\wedge_p} \cong \mathrm{Fil}_1^{\mathrm{conj}}(\overline{\Delta}_{X/A}),$$

where the right-hand side is the first piece of the increasing filtration on $\overline{\Delta}_{X/A}$ introduced in Proposition 3.1.11.

Proof. We can assume that $X = \mathrm{Spf}(R)$ is affine. We want to prove that there is a canonical isomorphism

$$\alpha_R : L_{R/A}\{-1\}[-1]^{\wedge_p} \cong \mathrm{Fil}_1^{\mathrm{conj}}(\overline{\Delta}_{R/A}).$$

First, let us note that by the transitivity triangle for $A \to A/I \to R$ the cotangent complex $L_{R/A}\{-1\}[-1]^{\wedge_p}$ sits inside a triangle

$$R \cong R \otimes_{A/I} L_{(A/I)/A}\{-1\}[-1]^{\wedge_p} \to L_{R/A}\{-1\}[-1]^{\wedge_p} \to L_{R/(A/I)}\{-1\}[-1]^{\wedge_p}$$

and the outer terms are isomorphic to $R \cong \mathrm{gr}_0^{\mathrm{conj}} \overline{\mathbbl{\Delta}}_{R/A}$ and

$$\mathrm{gr}_1^{\mathrm{conj}} \overline{\mathbbl{\Delta}}_{R/A} \cong L_{R/(A/I)}\{-1\}[-1]^{\wedge_p}.$$

To construct the isomorphism α_R it suffices to restrict to $A/I \to R$ p-completely smooth first, and then Kan extend. Thus assume from now on that R is p-completely smooth over A/I.

Let $(B, J) \in (R/A)_{\mathbbl{\Delta}}$, i.e., (B, J) is a prism over (A, I) with a morphism $\iota\colon R \to B/J$. Pulling back the extension of A-algebras

$$0 \to J/J^2 \to B/J^2 \to B/J \to 0$$

along $\iota\colon R \to B/J$ defines an extension of R by $J/J^2 \cong B/J\{1\}$ and as such, is thus classified by a morphism

$$\alpha'_R\colon L_{R/A}^{\wedge_p} \to B/J\{1\}[1].$$

Passing to the (homotopy) limit over all $(B, J) \in (R/A)_{\mathbbl{\Delta}}$ then defines (after shifting and twisting) the morphism

$$\alpha_R\colon L_{R/A}\{-1\}[-1]^{\wedge_p} \to \tau^{\leq 1} \overline{\mathbbl{\Delta}}_{R/A}.$$

Concretely, if $R = A/I\langle x \rangle$, then

$$L_{R/A}^{\wedge_p} \cong R \otimes_{A/I} I/I^2[1] \oplus R dx.$$

On the summand $R \otimes_{A/I} I/I^2[1]$, the morphism α'_R is simply the base extension of $I/I^2 \to J/J^2$ as follows by considering the case $A/I = R$. On the summand Rdx the morphism α'_R is (canonically) represented by the J/J^2-torsor of preimages of $\iota(x)$ in B/J^2 and factors as $R \xrightarrow{\iota} B/J \to B/J\{1\}[1]$ with the second morphism the connecting morphism for $0 \to B/J\{1\} \to B/J^2 \to B/J \to 0$. Thus, after passing to the limit, we get a diagram

$$\begin{tikzcd} R \arrow[d] \arrow[dr] & \\ \overline{\mathbbl{\Delta}}_{R/A} \arrow[r] & \overline{\mathbbl{\Delta}}_{R/A}\{1\}[1] \end{tikzcd}$$

and on H^0 the horizontal morphism induces the Bockstein differential

$$\beta\colon H^0(\overline{\mathbbl{\Delta}}_{R/A}) \to H^0(\overline{\mathbbl{\Delta}}_{R/A}\{1\}[1]) = H^1(\overline{\mathbbl{\Delta}}_{R/A})\{1\}.$$

Thus the image of $dx \in H^0(L_{R/A}^{\wedge_p})$ under α_R is $\beta(\iota(x))$. Therefore we see that on H^0 the morphism α_R induces the identity under the identifications

$$(\Omega^1_{R/(A/I)})^{\wedge_p} \cong H^0(L_{R/A}^{\wedge_p})$$

and

$$(\Omega^1_{R/(A/I)})^{\wedge_p} \cong H^1(\overline{\mathbbl{\Delta}}_{R/A})\{1\}$$

(the second is the Hodge-Tate comparison). Moreover, the morphism

$$R \otimes_{A/I} I/I^2 \cong H^{-1}(L_{R/A}^{\wedge_p}) \xrightarrow{H^{-1}(\alpha_R)} H^{-1}(\overline{\mathbbl{\Delta}}_{R/A}\{1\}[1])$$

is the canonical one obtained by tensoring $R \to H^0(\overline{\Delta}_{R/A})$ with I/I^2. By functoriality (and as $\Omega^1_{R/A}$ is generated by dr for $r \in R$), we can conclude that for every p-completely smooth algebra R over A

$$\alpha_R \colon H^i(L^{\wedge_p}_{R/A}) \to H^i(\overline{\Delta}_{R/A}\{1\}[1])$$

induces the canonical morphism, and thus, that α_R is an isomorphism in general. □

Recall the following proposition, which is a general consequence of the theory of the cotangent complex.

Proposition 3.2.2. *Let S be a ring, $I \subseteq S$ an invertible ideal and X a flat $\overline{S} := S/I$-scheme. Then the class $\gamma \in \mathrm{Ext}^2_{\mathcal{O}_X}(L_{X/\mathrm{Spec}(\overline{S})}, I/I^2 \otimes_{\overline{S}} \mathcal{O}_X)$ defined by $L_{X/\mathrm{Spec}(S)}$ is $\pm$ the obstruction class for lifting X to a flat S/I^2-scheme.*

Proof. See [24, III.2.1.2.3] resp. [24, III.2.1.3.3]. □

As before, let (A, I) be a bounded prism.

Corollary 3.2.3. *Let X be a p-completely flat p-adic formal scheme over A/I. The complex $\mathrm{Fil}^{\mathrm{conj}}_1 \overline{\Delta}_{X/A}$ splits in $D(X)$ (i.e., is isomorphic in $D(X)$ to a complex with zero differentials) if and only if X admits a lifting to a p-completely flat formal scheme over A/I^2.*

Proof. Indeed, $\mathrm{Fil}^{\mathrm{conj}}_1 \overline{\Delta}_{X/A}$ splits if and only if the class in

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathrm{gr}^{\mathrm{conj}}_1 \overline{\Delta}_{X/A}, \mathrm{gr}^{\mathrm{conj}}_0 \overline{\Delta}_{X/A}) = \mathrm{Ext}^2_{\mathcal{O}_X}(L^{\wedge_p}_{X/\mathrm{Spf}(A/I)}\{-1\}, \mathcal{O}_X)$$

defined by $\mathrm{Fil}^{\mathrm{conj}}_1(\overline{\Delta}_{X/A})$ vanishes. Proposition 3.2.1 shows that this class is the same as the class defined by the p-completed cotangent complex $L^{\wedge_p}_{X/\mathrm{Spf}(A)}\{-1\}$. Lifting X to a p-completely flat formal scheme over A/I^2 is the same as lifting $X \otimes_{A/I} A/(I, p^n)$ to a flat scheme over $A/(I^2, p^n)$ for all $n \geq 1$ (here we use that (A, I) is bounded in order to know that A/I is classically p-complete). One concludes by applying Proposition 3.2.2, together with the fact that the p-completion of the cotangent complex does not affect the (derived) reduction modulo p^n. □

This corollary will be used in Section 4.5, when studying prismatic cohomology of abelian schemes.

3.3. Quasi-syntomic rings.

We shortly recall some key definitions from [11, Chapter 4].

Definition 3.3.1. A ring R is *quasi-syntomic* if R is p-complete with bounded p^∞-torsion and if the cotangent complex $L_{R/\mathbb{Z}_p}$ has p-complete Tor-amplitude in $[-1, 0]$[11]. The category of all quasi-syntomic rings is denoted by QSyn.

Similarly, a map $R \to R'$ of p-complete rings with bounded p^∞-torsion is a *quasi-syntomic morphism* (resp. a *quasi-syntomic cover*) if R' is p-completely flat (resp. p-completely faithfully flat) over R and $L_{R'/R} \in D(R')$ has p-complete Tor-amplitude in $[-1, 0]$.

[11]This means that the complex $M = L_{R/\mathbb{Z}_p} \otimes^{\mathbb{L}}_R R/p \in D(R/p)$ is such that $M \otimes^{\mathbb{L}}_R N \in D^{[-1,0]}(R/p)$ for any R/p-module N.

For a quasi-syntomic ring R the p-completed cotangent complex $(L_{R/\mathbb{Z}_p})_p^\wedge$ will thus be in $D^{[-1,0]}$ (cf. [11, Lemma 4.6.]).

Remark 3.3.2. This definition extends (in the p-complete world) the usual notion of locally complete intersection ring and syntomic morphism (flat and local complete intersection) to the non-Noetherian, non finite-type setting, as shown by the next example.

Example 3.3.3. (1) Any p-complete l.c.i. Noetherian ring is in QSyn (cf. [2, Theorem 1.2]).
(2) There are also big rings in QSyn. For example, any (integral) perfectoid ring (i.e., a ring R which is p-complete, such that $\pi^p = pu$ for some $\pi \in R$ and $u \in R^\times$, Frobenius is surjective on R/p and $\ker(\theta)$ is principal.) is in QSyn (cf. [11, Proposition 4.18.]). We give a short explanation : if R is such a ring, the transitivity triangle for
$$\mathbb{Z}_p \to A_{\mathrm{inf}}(R) \to R$$
and the fact that $A_{\mathrm{inf}}(R)$ is relatively perfect over $\mathbb{Z}_p$ modulo p imply that after applying $-\otimes^{\mathbb{L}}_R R/p$, $L_{R/\mathbb{Z}_p}$ and $L_{R/A_{\mathrm{inf}}(R)}$ identify. But
$$L_{R/A_{\mathrm{inf}}(R)} = \ker(\theta)/\ker(\theta)^2[1] = R[1],$$
as $\ker(\theta)$ is generated by a non-zero divisor[12].
(3) As a consequence of (ii), the p-completion of a smooth algebra over a perfectoid ring is also quasi-syntomic, as well as any p-complete bounded p^∞-torsion ring which can be presented as the quotient of an integral perfectoid ring by a finite regular sequence.

The (opposite of the) category QSyn is endowed with the structure of a site.

Definition 3.3.4. Let $\mathrm{QSyn}^{\mathrm{op}}_{\mathrm{qsyn}}$ be the site whose underlying category is the opposite category of the category QSyn and endowed with the Grothendieck topology generated by quasi-syntomic covers.

If $R \in \mathrm{QSyn}$ we will denote by $(R)_{\mathrm{QSYN}}$ (resp. $(R)_{\mathrm{qsyn}}$) the big (resp. the small) quasi-syntomic site of R, given by all p-complete with bounded p^∞-torsion (resp. all quasi-syntomic) rings over R endowed with the quasi-syntomic topology).

The authors of [11] isolated an interesting class of quasi-syntomic rings.

Definition 3.3.5. A ring R is *quasi-regular semiperfectoid* if $R \in \mathrm{QSyn}$ and there exists a perfectoid ring S mapping surjectively to R.

Example 3.3.6. Any perfectoid ring, or any p-complete bounded p^∞-torsion quotient of a perfectoid ring by a finite regular sequence, is quasi-regular semiperfectoid.

The interest of quasi-regular semiperfectoid rings comes from the fact that they form a basis of the site $\mathrm{QSyn}^{\mathrm{op}}_{\mathrm{qsyn}}$.

Proposition 3.3.7. *Let R be quasi-syntomic ring. There exists a quasi-syntomic cover $R \to R'$, with R' quasi-regular semiperfectoid. Moreover, all terms of the Čech nerve $R'^{\bullet}$ are quasi-regular semiperfectoid.*

Proof. See [11, Lemma 4.27] and [11, Lemma 4.29]. □

[12] One also proves that $R[p^\infty] = R[p]$, which shows that R has bounded p^∞-torsion.

Finally, recall the following result, which is [12, Prop 7.11].

Proposition 3.3.8. *Let (A,I) be a bounded prism, and R be a quasi-syntomic A/I-algebra. There exists a prism $(B,IB)\in(R/A)_{\Delta}$ such that the map $R\to B/IB$ is p-completely faithfully flat. In particular, if $A/I\to R$ is a quasi-syntomic cover, then $(A,I)\to(B,IB)$ is a faithfully flat map of prisms.*

Proof. Since the proof is short, we recall it. Choose a surjection

$$A/I\langle x_j, j\in J\rangle\to R,$$

for some index set J. Set

$$S=A/I\langle x_j^{1/p^\infty}\rangle\hat{\otimes}^{\mathbb{L}}_{A/I\langle x_j,j\in J\rangle}R.$$

Then $R\to S$ is a quasi-syntomic cover and by assumption $A/I\to R$ is quasi-syntomic : hence, the map $A/I\to S$ is quasi-syntomic. Moreover the p-completion of $\Omega^1_{S/(A/I)}$ is zero. We deduce that the map $A/I\to S$ is such that $(L_{S/(A/I)})^{\wedge_p}$ has p-complete Tor-amplitude in degree $[-1,-1]$. Therefore, by the Hodge-Tate comparison, the derived prismatic cohomology $\Delta_{S/A}$ is concentrated in degree 0 and the map $S\to\overline{\Delta_{S/A}}$ is p-completely faithfully flat. One can thus just take $B=\Delta_{S/A}$. □

As observed in [12], a corollary of Proposition 3.3.8 is André's lemma.

Theorem 3.3.9 (André's lemma)**.** *Let R be perfectoid ring. Then there exists a p-completely faithfully flat map $R\to S$ of perfectoid rings such that S is absolutely integrally closed, i.e., every monic polynomial with coefficients in S has a solution.*

Proof. This is [12, Theorem 7.12]. Since the proof is also short, we recall it. Write $R=A/I$, for a perfect prism (A,I) (Proposition 2.1.8). The p-complete R-algebra $\tilde{R}$ obtained by adding roots of all possible monic polynomials over R is a quasi-syntomic cover, so by Proposition 3.3.8, we can find a prism (B,J) over (A,I) with a p-completely faithfully flat morphism $\tilde{R}\to R_1:=B/J$. Moreover, we can (and do) assume that (B,J) is a perfect prism, since going to the perfection[13] of a morphism of prisms preserves (p,I)-complete faithful flatness (because this can be checked modulo (p,I) and is true for the usual perfection on $\mathbb{F}_p$-algebras) and because (A,I) is already perfect. Transfinitely iterating the construction $R\mapsto R_1$ produces the desired ring S. □

Let us recall that a functor $u\colon\mathcal{C}\to\mathcal{D}$ between sites is cocontinuous (cf. [49, Tag 00XI]) if for every object $C\in\mathcal{C}$ and any covering $\{V_j\to u(C)\}_j$ of $u(C)$ in $\mathcal{D}$ there exists a covering $\{C_j\to C\}_j$ of C in $\mathcal{C}$ and morphisms $u(C_j)\to V_j$ over $u(C)$. For a cocontinuous functor $u\colon\mathcal{C}\to\mathcal{D}$ the functor

$$u^{-1}\colon\mathrm{Shv}(\mathcal{D})\to\mathrm{Shv}(\mathcal{C}),\ \mathcal{F}\to(\mathcal{F}\circ u)^\sharp$$

(here $()^\sharp$ denotes sheafification) is left-exact (even exact) with rightadjoint

$$\mathcal{G}\in\mathrm{Shv}(\mathcal{C})\mapsto(D\mapsto\varprojlim_{\{u(C)\to D\}^{\mathrm{op}}}\mathcal{G}(C)).$$

Thus, a cocontinuous functor $u\colon\mathcal{C}\to\mathcal{D}$ induces a morphism of topoi

$$u\colon\mathrm{Shv}(\mathcal{C})\to\mathrm{Shv}(\mathcal{D}).$$

[13]The *perfection* of a prism is the (p,I)-derived completion (or classical) of its colimit along φ. See [12].

Note that in the definition of a cocontinuous functor the morphisms $u(C_j) \to u(C)$ are not required to form a covering of C.

Corollary 3.3.10. *Let R be a p-complete ring. The functor $u\colon (R)_{\Delta} \to (R)_{\mathrm{QSYN}}$, sending (A, I) to*

$$R \to A/I$$

is cocontinuous. Consequently, it defines a morphism of topoi, still denoted by u :

$$u\colon \mathrm{Shv}((R)_{\Delta}) \to \mathrm{Shv}((R)_{\mathrm{QSYN}}).$$

Proof. Immediate from the definition (cf. [49, Tag 00XJ]) and the previous proposition. □

This yields the following important corollary.

Corollary 3.3.11. *Let R be a p-complete ring. Let*

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

be a short exact sequence of abelian sheaves on $(R)_{\mathrm{QSYN}}$. Then the sequence

$$0 \to u^{-1}(G_1) \to u^{-1}(G_2) \to u^{-1}(G_3) \to 0$$

is an exact sequence on $(R)_{\Delta}$. This applies for example when G_1, G_2, G_3 are finite locally free group schemes over R.

Proof. The first assertion is just saying that u^{-1} is exact, as u is a cocontinuous functor ([49, Tag 00XL]). The second assertion follows, as any finite locally free group scheme is syntomic (cf. [15, Proposition 2.2.2.]). □

3.4. **Prismatic cohomology of quasi-regular semiperfectoid rings.** In this section we want to explain why different natural choices of prismatic cohomology associated with a quasi-regular semiperfectoid ring are isomorphic. The same result can be found in [12]. We include this section since it offers a different argument.

For the moment, fix a bounded base prism (A, I) and let R be p-complete A/I-algebra. There are several cohomologies attached to R :

(1) The derived prismatic cohomology

$$\Delta_{R/A}$$

of R over (A, I) defined in Definition 3.1.9 via left Kan extension of prismatic cohomology.

(2) The cohomology

$$\Delta_{R/A}^{\mathrm{init}} := R\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta})$$

of the prismatic site of $(R/A)_{\Delta}$ (with its p-completely faithfully flat topology).

(3) Finally (and only for technical purposes),

$$\Delta_{R/A}^{\mathrm{init,unbdd}} := R\Gamma((R/A)_{\Delta,\mathrm{unbdd}}, \mathcal{O}_{\Delta}),$$

the prismatic cohomology of R with respect to the site $(R/A)_{\Delta,\mathrm{unbdd}}$ of not necessarily bounded prisms (B, J) over (A, I) together with a morphism $R \to B/J$ of A/I-algebras. We equip $(R/A)_{\Delta,\mathrm{unbdd}}$ with the chaotic topology.

These three cohomologies are different in general.

Lemma 3.4.1. *Assume that R is p-completely smooth over A/I. Then canonically*

$$\Delta_{R/A} \cong \Delta^{\mathrm{init}}_{R/A} \cong \Delta^{\mathrm{init,unbdd}}_{R/A}.$$

Proof. The first isomorphism follows from properties of the left Kan extension because R is p-completely smooth. For the second note that $\Delta^{\mathrm{init,unbdd}}_{R/A}$ can be computed via a Čech-Alexander complex as in [12, Construction 4.16] with all terms (p, I)-completely flat over A. This implies that all terms are bounded prisms (cf. [11, Corollary 4.8.]) and thus $\Delta^{\mathrm{init,unbdd}}_{R/A} \cong \Delta^{\mathrm{init}}_{R/A}$. □

Now we restrict our discussion to quasi-regular semiperfectoid rings. Assume from now on that (A, I) is a perfect prism and that $A/I \to R$ is a surjection with R quasi-regular semiperfectoid. The prism $\Delta^{\mathrm{init,unbdd}}_{R/A}$ admits then a more concrete (but in general rather untractable) description. Let K be the kernel of $A \to R$. Then

$$\Delta^{\mathrm{init,unbdd}}_{R/A} \cong A\{\frac{K}{I}\}^{\wedge_{(p,I)}}$$

is the prismatic envelope of the δ-pair (A, K) from [7, Lemma V.5.1] as follows from the universal property of the latter. In particular, the site $(R/A)_{\Delta,\mathrm{unbdd}}$ has a final object[14].

Proposition 3.4.2. *Let as above (A, I) be a perfect prism and R quasi-regular semiperfectoid with a surjection $A/I \twoheadrightarrow R$. Then the canonical morphisms induce isomorphisms*

$$\Delta_{R/A} \cong \Delta^{\mathrm{init}}_{R/A} \cong \Delta^{\mathrm{init,unbdd}}_{R/A}$$

as δ-rings.

Proof. For the second isomorphism it suffices to see that $\Delta^{\mathrm{init,unbb}}_{R/A}$ is bounded. For showing that $\Delta^{\mathrm{init,unbdd}}_{R/A}$ is bounded it suffices to prove that

$$\Delta_{R/A} \cong \Delta^{\mathrm{init,unbdd}}_{R/A}.$$

Namely, by the Hodge-Tate comparison $\Delta_{R/A}$ is bounded because the (shifted) cotangent complex $L^{\wedge}_{R/(A/I)}[-1]$ and all its shifted wedge powers have uniformly bounded p^∞-torsion. Namely they are p-completely flat over R (cf. [11, Lemma 4.7.]) and R is of bounded p^∞-torsion. By the universal property of left Kan extension (and Lemma 3.4.1) there exists a canonical morphism

$$\alpha_R \colon \Delta_{R/A} \to \Delta^{\mathrm{init,unbdd}}_{R/A}$$

compatible with the canonical morphisms $R \to \overline{\Delta}_{R/A}$ and $R \to \Delta^{\mathrm{init,unbdd}}_{R/A}$. Moreover, by the assumption that R is quasi-regular semiperfectoid the cotangent complex $L^{\wedge}_{R/(A/I)}$ is p-completely flat over R and concentrated in cohomological degree -1. By the Hodge-Tate comparison the complexes $\overline{\Delta}_{R/A}$ and $\Delta_{R/A}$ are therefore concentrated in degree 0. By [12, Lemma 7.7], $\Delta_{R/A}$ is canonically a δ-ring. Thus, using the universal property of $\Delta^{\mathrm{init,unbdd}}_{R/A}$, there is moreover a canonical morphism

$$\beta_R \colon \Delta^{\mathrm{init,unbdd}}_{R/A} \to \Delta_{R/A}.$$

[14] Up to now this discussion did not use that R is quasi-regular, it was sufficient that $A/I \to R$ is surjective.

The composition $\beta_R \circ \alpha_R = \mathrm{Id}_{\Delta_{R/A}}$ is the identity as follows from the universal property of left Kan extension. Thus it suffices to see that β_R is surjective. By (p, I)-adic completeness it suffices to check this for

$$\beta_R\colon \Delta_{R/A}^{\mathrm{init,unbdd}}/(p,I) \to \Delta_{R/A}/(p,I).$$

Let $(A, I) \to (A', I')$ be a faithfully flat morphism with (A', I') again perfect. Then

$$R' := A' \hat{\otimes}_A^{\mathbb{L}} R$$

is discrete and again quasi-regular semiperfectoid (here we use that R has bounded p^∞-torsion and [11, Lemma 4.6.]). Moreover,

$$\Delta_{R/A} \hat{\otimes}_A^{\mathbb{L}} A' \cong \Delta_{R'/A'}$$

as follows from the Hodge-Tate comparison and base change of the cotangent complex. From the universal property of $\Delta_{R/A}^{\mathrm{init,unbdd}}$ (and Proposition 2.1.11) we can deduce

$$H^0((A' \hat{\otimes}_A^{\mathbb{L}} \Delta_{R/A}^{\mathrm{init,unbdd}})^\wedge_{(p,I)}) \cong \Delta_{R'/A'}$$

where the completion is derived (p, I)-adic and we, a priori, have to take the H^0 as this completion might not be discrete as possibly $\Delta_{R/A}^{\mathrm{init,unbdd}}$ is unbounded. Moreover,

$$A'/(p,I) \otimes_A H^0((A' \hat{\otimes}_A^{\mathbb{L}} \Delta_{R/A}^{\mathrm{init,unbdd}})^\wedge_{(p,I)}) \cong A'/(p,I) \otimes_A \Delta_{R/A}^{\mathrm{init,unbdd}}.$$

Hence, it suffices to proof the claim for R' and (A', I') instead. By Theorem 3.3.9 we can therefore assume that the kernel $\overline{K}$ of $A/I \to R$ is generated by elements admitting p^n-roots for all $n \geq 0$. Set

$$S := A/I\langle X_k^{1/p^\infty} \mid k \in \overline{K}\rangle / X_j$$

and

$$B := A\langle X_k^{1/p^\infty} \mid k \in \overline{K}\rangle$$

where the completion is (p, I)-adic. After choosing compatible systems of p^n-roots for $\overline{k} \in \overline{K}$ and lifting this system to A, we obtain a morphism

$$B \to A,\ X_k^{1/p^n} \mapsto k^{1/p^n}$$

which in turn induces a morphism

$$S \to R.$$

By construction (and the Hodge-Tate comparison) the morphism

$$\overline{\Delta}_{S/B} \to \overline{\Delta}_{R/A}$$

is surjective as $\ker(B \to S) \to \overline{K}$ is surjective. Thus we can reduce the proof of surjectivity of β_R to S. As S is the completed colimit of quotients of perfectoid rings by regular sequences we can invoke [12, Ex. 7.9] to see that $\Delta_{R/A}^{\mathrm{init}} \cong \Delta_{R/A}$ in this case and [7, Lemma V.2.15] to see that $\Delta_{R/A}^{\mathrm{init,unbdd}} \cong \Delta_{R/A}^{\mathrm{init}}$ as it can be calculated via a Čech-Alexander complex all of whose terms are bounded (as they are (p, I)-completely flat over A). $\square$

If $pR = 0$, i.e., R is quasi-regular semiperfect, there is moreover the universal p-complete PD-thickening

$$A_{\mathrm{crys}}(R)$$

of R (cf. [48, Proposition 4.1.3]). The ring $A_{\mathrm{crys}}(R)$ is p-torsion free by [11, Theorem 8.14.].

Lemma 3.4.3. *Let (A, I), R be as above and assume that $pR = 0$. Then there is a canonical φ-equivariant isomorphism*

$$\mathbb{\Delta}_{R/A} \cong A_{\mathrm{crys}}(R).$$

Proof. As $A_{\mathrm{crys}}(R)$ is p-torsion free (cf. [11, Theorem 8.14]) and carries a canonical Frobenius lift there we get a natural morphism

$$\mathbb{\Delta}_{R/A} \to A_{\mathrm{crys}}(R).$$

Conversely, the kernel of the natural morphism (cf. Theorem 3.4.6, which does not depend on this lemma)

$$\theta \colon \mathbb{\Delta}_{R/A} \to R$$

has divided powers (as one checks similarly to [11, Proposition 8.12], using that the proof of Theorem 3.1.7 goes through in the syntomic case, cf. Remark 3.1.8). This provides a canonical morphism

$$A_{\mathrm{crys}}(R) \to \mathbb{\Delta}_R$$

in the other direction. Similarly, to [11, Theorem 8.14] one checks that both are inverse to each other. □

Remark 3.4.4. Both rings $\mathbb{\Delta}_{R/A}$ and $A_{\mathrm{crys}}(R)$ are naturally $W(R^\flat)$-algebras, but the isomorphism of Lemma 3.4.3 restricts to the Frobenius on $W(R^\flat)$. Concretely, if $R = R^\flat/x$ for some non-zero divisor $x \in R^\flat$, then

$$\mathbb{\Delta}_{R/W(R^\flat)} \cong W(R^\flat)\{\frac{x}{p}\}^\wedge$$

and (cf. [12, Corollary 2.37])

$$A_{\mathrm{crys}}(R) \cong W(R^\flat)\{\frac{x^p}{p}\}^\wedge \cong \mathbb{\Delta}_{R/W(R^\flat)} \otimes_{W(R^\flat),\varphi} W(R^\flat).$$

Finally, we will need to connect $\mathbb{\Delta}_R$ to topological cyclic homology. Let

$$\mathrm{TC}^-(R)$$

be the p-completed negative topological cyclic homology of R (cf. [11] and [43]). On $\pi_0(\mathrm{TC}^-(R))$ there is a canonical ring endomorphism $\varphi^{h\mathrm{S}^1}$ induced from the cyclotomic Frobenius $\varphi\colon \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$ on the (p-completed) topological Hochschild homology. After choosing a perfectoid ring $S \to R$ mapping to R there is a canonical morphism

$$R \to \pi_0(\mathrm{TC}^-(R))/\tilde{\xi}$$

(using that $\pi_0(\mathrm{TC}^-(R)) \cong \pi_0(\mathrm{TP}(R))$ and [11, Proposition 6.4.]).

Theorem 3.4.5. *The ring $\pi_0(\mathrm{TC}^-(R))$ is $(p, \tilde{\xi})$-adically complete, p-torsion free and the endomorphism $\varphi\colon \pi_0(\mathrm{TC}^-(R)) \to \pi_0(\mathrm{TC}^-(R))$ is a Frobenius lift. The induced morphism*

$$\mathbb{\Delta}_R \to \pi_0(\mathrm{TC}^-(R))$$

identifies the latter with the Nygaard completion $\hat{\mathbb{\Delta}}_R$.

Proof. See [12, Theorem 13.1]. □

In particular, one derives for $i \geq 0$ isomorphisms

$$\pi_{2i}(\mathrm{TC}(R)) \cong \hat{\mathbb{\Delta}}_R^{\varphi=\tilde{\xi}^i}$$

for the topological cyclic homology $\mathrm{TC}(R)$ of R (cf. [11, Section 7.4.]).

Finally, we recall the following statement from [12], identifying the associated gradeds of the Nygaard filtration.

Theorem 3.4.6. *Let R be a quasi-regular semiperfectoid ring. Then*

$$\mathcal{N}^{\geq i}(\mathbb{\Delta}_R)/\mathcal{N}^{\geq i+1}(\mathbb{\Delta}_R) \cong \mathrm{Fil}_i^{\mathrm{conj}}(\overline{\mathbb{\Delta}}_R)\{i\}$$

for $i \geq 0$. In particular, $\mathbb{\Delta}_R/\mathcal{N}^{\geq 1}\mathbb{\Delta}_R \cong R$.

Here $\mathrm{Fil}_\bullet^{\mathrm{conj}}(\overline{\mathbb{\Delta}}_R)$ denotes the conjugate filtration on $\overline{\mathbb{\Delta}}_R$ with graded pieces given by $\mathrm{gr}_i^{\mathrm{conj}}(\overline{\mathbb{\Delta}}_R) \cong \Lambda^i L_{R/\mathbb{Z}_p}^{\wedge_p}[-i]$.

Proof. See [12, Theorem 12.2]. □

3.5. **The Künneth formula in prismatic cohomology.** The Hodge-Tate comparison implies a Künneth formula. Here is the precise statement. Note that for a bounded prism (A, I) the functor $R \mapsto \mathbb{\Delta}_{R/A}$ is naturally defined on all derived p-complete simplicial A/I-algebras.

Proposition 3.5.1. *Let (A, I) be a bounded prism. Then the functor*

$$R \mapsto \mathbb{\Delta}_{R/A}$$

from derived p-complete simplicial rings over A/I to derived (p, I)-complete E_∞-algebras over A preserves tensor products, i.e., for all morphism $R_1 \leftarrow R_3 \rightarrow R_2$ the canonical morphism

$$\mathbb{\Delta}_{R_1/A}\hat{\otimes}^{\mathbb{L}}_{\mathbb{\Delta}_{R_3/A}}\mathbb{\Delta}_{R_2/A} \to \mathbb{\Delta}_{R_1\hat{\otimes}^{\mathbb{L}}_{R_3}R_2/A}$$

is an equivalence.

Proof. Using [11, Construction 2.1] (resp. [39, Proposition 5.5.8.15.]) the functor $R \mapsto \mathbb{\Delta}_{R/A}$, which is the left Kan extension from p-completely smooth algebras to all derived p-complete simplicial A/I-algebras, commutes with colimits if it preserves finite coproducts. Clearly, $\mathbb{\Delta}_{(A/I)/A} \cong A$, i.e., $\mathbb{\Delta}_{-/A}$ preserves the final object. Moreover, for R, S p-completely smooth over A/I the canonical morphism

$$\mathbb{\Delta}_{R/A}\hat{\otimes}^{\mathbb{L}}_A\mathbb{\Delta}_{S/A} \to \mathbb{\Delta}_{S\hat{\otimes}R/A}$$

is an isomorphism because this may by I-completeness be checked for $\overline{\mathbb{\Delta}}_{-/A}$ where it follows from the Hodge-Tate comparison. □

Gluing the isomorphism in Proposition 3.5.1 we can derive, using as well the projection formula and flat base change for quasi-coherent cohomology, the following statement.

Corollary 3.5.2. *If X and Y are proper, p-completely smooth p-adic formal schemes over $\mathrm{Spf}(A/I)$), then*

$$R\Gamma(X \times_{\mathrm{Spf}(A/I)} Y, \mathbb{\Delta}_{X\times_{\mathrm{Spf}(A/I)}Y/A}) \cong R\Gamma(X, \mathbb{\Delta}_{X/A})\hat{\otimes}^{\mathbb{L}}_A R\Gamma(Y, \mathbb{\Delta}_{Y/A}).$$

3.6. **A torsion-freeness result for prismatic cohomology.** Let C be a complete algebraically closed extension of $\mathbb{Q}_p$. Fix a choice $\varepsilon \in \mathcal{O}_{C^\flat}$ of compatible primitive p^n-th roots of unity and let $\mu = [\varepsilon]-1$. In this paragraph, we establish μ-torsion freeness of the prismatic cohomology of certain quasi-regular semiperfectoid rings over $\mathcal{O}_C/p^n$ (Proposition 3.6.2). Moreover, we set

$$(A, I) := (A_{\mathrm{inf}}(\mathcal{O}_C), \ker(\tilde{\theta}))$$

and fix a generator $\tilde{\xi} : 1 + q + \ldots + q^{p-1}$ of $\ker(\tilde{\theta})$.

Lemma 3.6.1. *The derived prismatic cohomology*

$$\mathbb{\Delta}_{(\mathbb{Z}/p^n \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{F}_p)/\mathbb{Z}_p}$$

is concentrated in degree 0 *for* $n \geq 1$.

Proof. Let P be a polynomial $\mathbb{Z}_p$-algebra. Then by the crystalline comparison, Theorem 3.1.7, and the comparison of crystalline cohomology with p-completed de Rham cohomology of a smooth lift, cf. [4, Theorem V.2.3.2],

$$\mathbb{\Delta}_{(P/p)/\mathbb{Z}_p} \cong R\Gamma_{\mathrm{crys}}((P/p)/\mathbb{Z}_p) \cong \mathrm{dR}^{\wedge_p}_{P/\mathbb{Z}_p}$$

Left Kan extending this isomorphism (in the category of derived p-complete complexes) yields an isomorphism

$$\mathbb{\Delta}_{S \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{F}_p/\mathbb{Z}_p} \cong \mathrm{dR}^{\wedge_p}_{S/\mathbb{Z}_p}$$

for any simplical $\mathbb{Z}_p$-algebra (to identify the left-hand side, we use that the left Kan extension of the functor sending a polynomial $\mathbb{Z}_p$-algebra P to the prismatic cohomology of P/p over $\mathbb{Z}_p$ is the same thing, by composition of left Kan extensions, as the composition of the derived tensor product functor $-\otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{F}_p$ with the left Kan extension of the functor sending a polynomial $\mathbb{F}_p$-algebra to its prismatic cohomology over $\mathbb{Z}_p$, which is by definition derived prismatic cohomology). In particular,

$$\mathbb{\Delta}_{\mathbb{Z}/p^n \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{F}_p} \cong \mathrm{dR}^{\wedge_p}_{(\mathbb{Z}/p^n)/\mathbb{Z}_p}.$$

By [8, Proposition 8.5],

$$\mathrm{dR}^{\wedge_p}_{(\mathbb{Z}/p^n)/\mathbb{Z}_p} \cong (D_{\mathbb{Z}[x]}((x))^{\wedge_p} \xrightarrow{x-p^n} D_{\mathbb{Z}[x]}((x))^{\wedge_p})$$

with the right hand side sitting in degrees $-1, 0$. As the p-completed free divided power algebra $D_{\mathbb{Z}[x]}((x))^{\wedge_p}$ is an integral domain, we can conclude that as desired

$$\mathrm{dR}^{\wedge_p}_{(\mathbb{Z}/p^n)/\mathbb{Z}_p}$$

is concentrated in degree 0. □

Proposition 3.6.2. *Let* $n \geq 1$. *Let* R *be quasi-regular semiperfectoid and flat over* $\mathbb{Z}/p^n$. *Define* $R' := R \otimes_{\mathbb{Z}/p^n} \mathcal{O}_C/p^n$. *Then*

$$\mathbb{\Delta}_{R'}$$

is μ*-torsionfree.*

Proof. We have to show that

$$\mathbb{\Delta}_{R'} \otimes^{\mathbb{L}}_A A/\mu$$

is concentrated on degree 0. The morphism $A \to A/\mu$ factors over A_{crys} by Lemma 3.6.3 below. Thus what we want to prove is that

$$(\Delta_{R'}\hat{\otimes}^{\mathbb{L}}_{A} A_{\mathrm{crys}})\hat{\otimes}^{\mathbb{L}}_{A_{\mathrm{crys}}} A/\mu$$

is concentrated in degree 0. The natural map $A \to A_{\mathrm{crys}}$ gives rise to a morphism of prisms

$$(A, (\tilde{\xi})) \to (A_{\mathrm{crys}}, (p))$$

and thus by base change for derived prismatic cohomology

$$\Delta_{R'}\hat{\otimes}^{\mathbb{L}}_{A} A_{\mathrm{crys}} \cong \Delta_{R'\hat{\otimes}^{\mathbb{L}}_{\mathcal{O}_C} A_{\mathrm{crys}}/p}$$

We can calculate

$$\begin{aligned} R' \otimes^{\mathbb{L}}_{\mathcal{O}_C} A_{\mathrm{crys}}/p &\cong R \otimes^{\mathbb{L}}_{\mathbb{Z}_p} A_{\mathrm{crys}}/p \\ &\cong R \otimes^{\mathbb{L}}_{\mathbb{Z}/p^n} (\mathbb{Z}/p^n \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{F}_p) \otimes^{\mathbb{L}}_{\mathbb{F}_p} A_{\mathrm{crys}}/p \\ &\cong (R \otimes^{\mathbb{L}}_{\mathbb{Z}/p^n} \mathbb{F}_p) \otimes^{\mathbb{L}}_{\mathbb{F}_p} (\mathbb{Z}/p^n \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{F}_p) \otimes^{\mathbb{L}}_{\mathbb{F}_p} A_{\mathrm{crys}}/p \\ &\cong R/p \otimes^{\mathbb{L}}_{\mathbb{F}_p} (\mathbb{Z}/p^n \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{F}_p) \otimes^{\mathbb{L}}_{\mathbb{F}_p} A_{\mathrm{crys}}/p. \end{aligned}$$

The first isomorphism follows from the definition of R' (and $\mathcal{O}_C/p^n \cong \mathbb{Z}/p^n \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathcal{O}_C$ by p-torsion freeness of $\mathcal{O}_C$), in the second isomorphism we inserted some factors while in the third isomorphism we use that the canonical morphism $\mathbb{Z}/p^n \to \mathbb{Z}/p^n \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{F}_p$ factors (in the derived category) over $\mathbb{F}_p$[15]. The fourth isomorphism follows from flatness of R over $\mathbb{Z}/p^n$. Thus,

$$\Delta_{R'\otimes^{\mathbb{L}}_{\mathcal{O}_C} A_{\mathrm{crys}}/p/A_{\mathrm{crys}}} \cong \Delta_{R/p}\hat{\otimes}^{\mathbb{L}}_{\mathbb{Z}_p}\Delta_{\mathbb{Z}/p^n\otimes^{\mathbb{L}}_{\mathbb{Z}_p}\mathbb{F}_p/\mathbb{Z}_p}\hat{\otimes}^{\mathbb{L}}_{\mathbb{Z}_p} A_{\mathrm{crys}}$$

using base change along the morphism $(\mathbb{Z}_p, (p)) \to (A_{\mathrm{crys}}, (p))$ and the Künneth formula Proposition 3.5.1. Our aim is therefore to prove that

$$\Delta_{R/p}\hat{\otimes}^{\mathbb{L}}_{\mathbb{Z}_p}\Delta_{\mathbb{Z}/p^n\otimes^{\mathbb{L}}_{\mathbb{Z}_p}\mathbb{F}_p/\mathbb{Z}_p}\hat{\otimes}^{\mathbb{L}}_{\mathbb{Z}_p} A/\mu$$

is concentrated in degree 0.

The rings A/μ and $\Delta_{R/p}$ are p-torsion free (the latter since the ring R/p is quasi-regular semiperfect : it is obviously semiperfect, and is also quasi-syntomic by [11, Lemma 4.15 (2)]). Hence, they are topologically free as $\mathbb{Z}_p$-modules[16]. It therefore suffices to show that the derived prismatic cohomology $\Delta_{\mathbb{Z}/p^n\otimes^{\mathbb{L}}_{\mathbb{Z}_p}\mathbb{F}_p/\mathbb{Z}_p}$ of the simplicial ring $\mathbb{Z}/p^n \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{F}_p$ over $\mathbb{F}_p$ is concentrated in degree 0. This is the content of Lemma 3.6.1. Note that we can not apply the Künneth formula to $\Delta_{\mathbb{Z}/p^n\otimes^{\mathbb{L}}_{\mathbb{Z}_p}\mathbb{F}_p/\mathbb{Z}_p}$ as the tensor product is taken over $\mathbb{Z}_p$. □

Lemma 3.6.3. *The morphism $A \to A/\mu$ factors uniquely over A_{crys}.*

Proof. We have $\varphi(\mu) = \tilde{\xi}\mu$ and thus the (derived) p-complete p-torsion free ring A/μ is naturally a δ-ring[17]. As

$$A_{\mathrm{crys}} \cong A_{\mathrm{inf}}\{\frac{\tilde{\xi}}{p}\}^{\wedge_p}$$

[15]This canonical morphism is represented by the morphism of complexes $(\mathbb{Z}_p \xrightarrow{p^n} \mathbb{Z}_p) \to (\mathbb{Z}/p^n \xrightarrow{p} \mathbb{Z}/p^n)$. Now it is clear that this morphism factors over $\mathbb{F}_p \cong (\mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p)$.

[16]Therefore, the problem of Remark 4.8.7 does not appear in this case.

[17]By [10, Lemma 3.23], $A/\mu \hookrightarrow W(\mathcal{O}_C)$ with equality if C is spherically complete.

by [12, Lemma 2.35] it suffices to see that $\tilde{\xi}$ is divisible by p in A/μ. By p-torsionfreeness of A/μ this implies uniqueness. But clearly

$$\tilde{\xi} = 1 + q + \ldots + q^{p-1} \equiv p \bmod \mu$$

as $\mu = q - 1$. □

Example 3.6.4. Proposition 3.6.2 is not a direct consequence of a torsion-freeness statement for flat $\mathcal{O}_C/p^n$-modules, as we now illustrate. Let us construct an example of a ring R which is quasi-regular semiperfectoid and flat over $\mathcal{O}_C/p$ such that Δ_R contains no almost zero elements, i.e., elements $x \in \Delta_R$ such that $W(\mathfrak{m}^\flat)x = 0$ where $\mathfrak{m}^\flat \subseteq \mathcal{O}_C^\flat$ is the maximal ideal, but $\overline{\Delta}_R$ does. Let $\mathfrak{m} \subseteq \mathcal{O}_C$ be the maximal ideal. The flat $\mathcal{O}_C/p$-module

$$\mathfrak{m} \otimes_{\mathcal{O}_C} \mathcal{O}_C/p \cong \mathfrak{m}/p\mathfrak{m}$$

contains the non-zero almost zero element p. Lifting this example to the world of quasi-regular semiperfectoid rings will provide our example. Let

$$\tilde{S} \subseteq \mathcal{O}_C\langle X^{1/p^\infty}\rangle$$

be the subring of elements $\sum\limits_{i \in \mathbb{Z}[1/p]_{\geq 0}} a_i X^i$ such that $a_i \in \mathfrak{m}$ for $i > 0$. Set

$$\tilde{R} := \tilde{S}/\mathfrak{m}X.$$

Then $\tilde{R}$ is quasi-regular semiperfectoid. Let

$$p^\flat := (p, p^{1/p}, p^{1/p^2}, \ldots)$$

be a compatible system of p^m-roots of p. As

$$\mathfrak{m} = \bigcup_{m \geq 0} (p^{1/p^m})$$

we can write

$$\tilde{R} = \varinjlim_{m \geq 0} \mathcal{O}_C\langle X_m^{1/p^\infty}\rangle/(X_m)$$

where the transition maps send X_m^{1/p^i} to $((p^{1/p^{m+1}})^{p-1})^{1/p^i} X_{m+1}^{1/p^i}$[18]. This allows us to compute $\Delta_{\tilde{R}}$ using [12, Lemma 12.3.]

$$\Delta_{\mathcal{O}_C\langle X_m^{1/p^\infty}\rangle/(X_m)} \cong \Big(\bigoplus_{i \in \mathbb{Z}[1/p]} A \frac{Y_m^i}{[i]_q!}\Big)^{\wedge (p,q-1)}$$

with $q = [\varepsilon]$ the Teichmüller of a compatible system of p^j-roots of unity and $Y_m = X_m^{1/p}$. Passing to the colimit over m implies that

$$\Delta_{\tilde{R}} \cong \Big(A \oplus \bigoplus_{i \in \mathbb{Z}[1/p]_{>0}} W(\mathfrak{m})^\flat \frac{Y_m^i}{[i]_q!}\Big)^{\wedge (p,q-1)}.$$

Finally, set

$$R := \tilde{R}/p.$$

Then

$$\Delta_R \cong \Delta_{\mathcal{O}_C/p} \hat{\otimes}_A^{\mathbb{L}} \Delta_{\tilde{R}}$$

[18] More precisely, we set $X_m^{1/p^i} = p^{1/p^{m}\,1/p^i} X^{1/p^i}$.

by Proposition 3.5.1. Thus, to show that Δ_R contains no non-zero almost zero elements it suffices to show that

$$B\hat{\otimes}^{\mathbb{L}}_A W(\mathfrak{m}^\flat)$$

contains no non-zero almost zero elements, where $B := \Delta_{\mathcal{O}_C/p}$. By definition there is an exact sequence

$$0 \to W(\mathfrak{m}^\flat) \to A \to W(k) \to 0$$

where $k = \mathcal{O}_C/\mathfrak{m}$ is the residue field of $\mathcal{O}_C$. It suffices to show that

$$B\hat{\otimes}^{\mathbb{L}}_A W(\mathfrak{m}^\flat) \to B$$

is injective as B contains no non-zero almost zero elements[19]. For this it suffices to see that $B\hat{\otimes}^{\mathbb{L}}_A W(k)$ is concentrated in degree 0. But

$$B \cong D_{\mathbb{Z}[x]}((x))\hat{\otimes}^{\mathbb{L}}_{\mathbb{Z}[x]} A$$

where $\mathbb{Z}[x] \to A,\ x \mapsto \xi$, with ξ a distinguished element reducing to p in $W(k)$. The morphism $\mathbb{Z}[x] \to A$ is a flat as follows from [10, Remark 4.31], which implies that the above tensor product is concentrated in degree 0. Therefore,

$$B\hat{\otimes}^{\mathbb{L}}_A W(k) \cong D_{\mathbb{Z}[x]}((x))\hat{\otimes}^{\mathbb{L}}_{\mathbb{Z}[x]} W(k).$$

We may replace $W(k)$ by $\mathbb{Z}_p$ by faithful flatness of $\mathbb{Z}_p \to W(k)$. Finally,

$$D_{\mathbb{Z}[x]}((x))\hat{\otimes}^{\mathbb{L}}_{\mathbb{Z}[x]} \mathbb{Z}_p \cong (D_{\mathbb{Z}[x]}((x)) \xrightarrow{x-p} D_{\mathbb{Z}[x]}((x)))$$

is concentrated in degree 0 as the divided power algebra $D_{\mathbb{Z}[x]}((x))$ is an integral domain. This implies desired that Δ_R has no non-zero almost zero elements. However, $\overline{\Delta}_R$ has non-zero almost zero elements as R has (and R embeds into $\overline{\Delta}_R$).

[19]As B is p-complete and p-torsion free this reduces to the same statement over B/p, which is free over $\mathcal{O}_C/p$.

4. Prismatic Dieudonné theory for p-divisible groups

This chapter is the heart of this paper. We construct the *(filtered) prismatic Dieudonné functor* over any quasi-syntomic ring and prove that it gives an antiequivalence between p-divisible groups over R and *filtered prismatic Dieudonné crystals over* R, for quasi-syntomic rings R which are flat over $\mathbb{Z}_p$ or over $\mathbb{Z}/p^n$ for some $n \geq 1$. The strategy to do this is to use *quasi-syntomic descent* to reduce to the case where R is quasi-regular semiperfectoid, in which case the filtered prismatic Dieudonné crystals over R can be replaced by simpler objects, the *filtered prismatic Dieudonné modules.*

4.1. **Abstract filtered prismatic Dieudonné crystals and modules.**

Let R be a p-complete ring. We defined in Corollary 3.3.10 a morphism of topoi :

$$u : \mathrm{Shv}((R)_{\Delta}) \to \mathrm{Shv}((R)_{\mathrm{QSYN}}).$$

We let v be the composite of u with the morphism of topos induced by restriction to the small quasi-syntomic site $(R)_{\mathrm{qsyn}}$ of R, formed by rings which are quasi-syntomic over R, endowed with the quasi-syntomic topology.

Definition 4.1.1. Let R be a p-complete ring. We define :

$$\mathcal{O}^{\mathrm{pris}} := v_*\mathcal{O}_{\Delta} \ ; \ \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}} := v_*\mathcal{N}^{\geq 1}\mathcal{O}_{\Delta} \ ; \ \mathcal{I}^{\mathrm{pris}} := v_*\mathcal{I}_{\Delta},$$

where $\mathcal{I}_{\Delta} \subseteq \mathcal{O}_{\Delta}$ denotes the canonical invertible ideal sheaf sending a prism $(B, J) \in (R)_{\Delta}$ to J. The sheaf $\mathcal{O}^{\mathrm{pris}}$ is endowed with a Frobenius lift φ.

Although these sheaves are defined in general, we will only use them over quasi-syntomic rings.

Proposition 4.1.2. *Let R be quasi-syntomic ring. The quotient sheaf*

$$\mathcal{O}^{\mathrm{pris}}/\mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}$$

is isomorphic to the structure sheaf $\mathcal{O}$ of $(R)_{\mathrm{qsyn}}$.

Proof. It is enough to produce such an isomorphism functorially on a basis of $(R)_{\mathrm{qsyn}}$. By Proposition 3.3.7, we can thus assume that R is quasi-regular semiperfectoid. In this case, we conclude by Theorem 3.4.6. □

Definition 4.1.3. Let R be a p-complete ring. A *prismatic crystal* over R is an $\mathcal{O}_{\Delta}$-module $\mathcal{M}$ on the prismatic site $(R)_{\Delta}$ of R such that for all morphisms $(B, J) \to (B', J')$ in $(R)_{\Delta}$ the canonical morphism

$$\mathcal{M}(B, J) \otimes_B B' \cong \mathcal{M}(B', J')$$

Note that a prismatic crystal in finite locally free $\mathcal{O}_{\Delta}$-modules (resp. in finite locally free $\overline{\mathcal{O}}_{\Delta}$-modules) is the same thing as a finite locally free $\mathcal{O}_{\Delta}$-module (resp. a finite locally free $\overline{\mathcal{O}}_{\Delta}$-module). In what follows, we will essentially consider only this kind of prismatic crystals.

Proposition 4.1.4. *Let R be a quasi-syntomic ring. The functors v_* and $v^*(-) := \mathcal{O}_{\Delta} \otimes_{v^{-1}\mathcal{O}^{\mathrm{pris}}} v^{-1}$ induce equivalences between the category of finite locally free $\mathcal{O}_{\Delta}$-modules and the category of finite locally free $\mathcal{O}^{\mathrm{pris}}$-modules.*

Proof. Because $v_*(\mathcal{O}_{\mathbb{\Delta}}) = \mathcal{O}^{\mathrm{pris}}$ it is clear that for all finite locally free $\mathcal{O}^{\mathrm{pris}}$-modules $\mathcal{M}$ the canonical morphism

$$\mathcal{M} \to v_*(v^*(\mathcal{M}))$$

is an isomorphism as this can be checked locally on $(R)_{\mathrm{qsyn}}$. Conversely, let $\mathcal{N}$ be a finite locally free $\mathcal{O}_{\mathbb{\Delta}}$-module. We have to show that the counit

$$v^*v_*(\mathcal{N}) \to \mathcal{N}$$

is an isomorphism. For any morphism $R \to R'$ with R' quasi-syntomic there are equivalences

$$(R)_{\mathbb{\Delta}}/h_{R'} \cong (R')_{\mathbb{\Delta}}\ ,\ (R)_{\mathrm{qsyn}}/R' \cong (R')_{\mathrm{qsyn}}$$

of slice topoi where $h_{R'}(B, J) := \mathrm{Hom}_R(R', B/J)$. By passing to a quasi-syntomic cover $R \to R'$ we can therefore assume that R is quasi-regular semiperfectoid, in particular that the site $(R)_{\mathbb{\Delta}}$ has an inital object given by $\mathbb{\Delta}_R$. By (p, I)-completely faithfully flat descent of finite locally free modules over (p, I)-complete rings of bounded (p, I)-torsion (cf. Proposition A.12), the category of finite locally free $\mathcal{O}_{\mathbb{\Delta}}$-modules on $(R)_{\mathbb{\Delta}}$ is equivalent to finite locally free $\mathbb{\Delta}_R$-modules[20]. As the morphism $\mathbb{\Delta}_R \to R$ (the "θ"-map) is henselian along its kernel, cf. Lemma 4.1.24, finite locally free $\mathbb{\Delta}_R$-modules split on the pullback of an open cover of $\mathrm{Spf}(R)$. Thus, after passing to a quasi-syntomic R-algebra, we may assume that $\mathcal{N}$ is finite free. Then the isomorphism

$$v^*v_*(\mathcal{N}) \cong \mathcal{N}$$

is clear. $\square$

Definition 4.1.5. Let R be a quasi-syntomic ring. A *prismatic Dieudonné crystal over R* is a finite locally free $\mathcal{O}^{\mathrm{pris}}$-module $\mathcal{M}$ together with φ-linear morphism

$$\varphi_{\mathcal{M}} \colon \mathcal{M} \to \mathcal{M}$$

whose linearization has its cokernel is killed by $\mathcal{I}^{\mathrm{pris}}$.

Definition 4.1.6. Let R be a quasi-syntomic ring. A *filtered prismatic Dieudonné crystal over R* is a collection $(\mathcal{M}, \mathrm{Fil}\mathcal{M}, \varphi_{\mathcal{M}})$ consisting of a finite locally free $\mathcal{O}^{\mathrm{pris}}$-module $\mathcal{M}$, a $\mathcal{O}^{\mathrm{pris}}$-submodule $\mathrm{Fil}\mathcal{M}$, and a φ-linear map $\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$, satisfying the following conditions :

(1) $\varphi_{\mathcal{M}}(\mathrm{Fil}\mathcal{M}) \subset \mathcal{I}^{\mathrm{pris}}.\mathcal{M}$.
(2) $\mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}.\mathcal{M} \subset \mathrm{Fil}\mathcal{M}$ and $\mathcal{M}/\mathrm{Fil}\mathcal{M}$ is a finite locally free $\mathcal{O}$-module.
(3) $\varphi_{\mathcal{M}}(\mathrm{Fil}\mathcal{M})$ generates $\mathcal{I}^{\mathrm{pris}}.\mathcal{M}$ as an $\mathcal{O}^{\mathrm{pris}}$-module.

For a prismatic Dieudonné crystal $(\mathcal{M}, \varphi_{\mathcal{M}})$ the linearization of the morphism $\varphi_{\mathcal{M}} \colon \mathcal{M} \to \mathcal{M}$ is an isomorphism after inverting a local generator $\tilde{\xi}$ of $\mathcal{I}^{\mathrm{pris}}$ and in particular is injective, since $\varphi^*\mathcal{M}$ is $\tilde{\xi}$-torsion free.

The last condition in Definition 4.1.6 implies in particular that if $(\mathcal{M}, \mathrm{Fil}\,\mathcal{M}, \varphi_{\mathcal{M}})$ is a filtered prismatic Dieudonné crystal over R, then $(\mathcal{M}, \varphi_{\mathcal{M}})$ is a prismatic Dieudonné crystal over R.

Definition 4.1.7. Let R be a quasi-syntomic ring. We denote by $\mathrm{DM}(R)$ the category of prismatic Dieudonné crystals over R (with $\mathcal{O}^{\mathrm{pris}}$-linear morphisms commuting with Frobenius) and by $\mathrm{DF}(R)$ the category of filtered prismatic Dieudonné

[20]The non-trivial point is that the global sections of a finite locally free $\mathcal{O}_{\mathbb{\Delta}}$-module are locally free over $\mathbb{\Delta}_R$.

crystals over R (with morphisms $\mathcal{O}^{\mathrm{pris}}$-linear morphisms commuting with the Frobenius and respecting the filtration).

Proposition 4.1.8. *The fibered category of filtered prismatic Dieudonné crystals over the category* QSyn *of quasi-syntomic rings endowed with the quasi-syntomic topology is a stack.*

Proof. This follows from the definition, because by general properties of topoi modules under $\mathcal{O}^{\mathrm{pris}}$ and $\mathcal{O}$ form a stack for the quasi-syntomic topology on $(R)_{\mathrm{qsyn}}$. $\square$

For quasi-regular semiperfectoid rings, these abstract objects have a more concrete incarnation, which we explain now. Let R be a quasi-regular semiperfectoid ring and let (Δ_R, I) be the prism associated with R. Note that I is necessarily principal as there exists a perfectoid ring mapping to R. Recall (Theorem 3.4.6) that

$$\theta\colon \Delta_R/\mathcal{N}^{\geq 1}\Delta_R \cong R$$

is an isomorphism.

Definition 4.1.9. A *prismatic Dieudonné module over* R is a finite locally free Δ_R-module M together with a φ-linear morphism

$$\varphi_M\colon M \to M$$

whose linearization has its cokernel is killed by I.

Definition 4.1.10. A *filtered prismatic Dieudonné module over* R is a collection $(M, \mathrm{Fil}\ M, \varphi_M)$ consisting of a finite locally free Δ_R-module M, a Δ_R-submodule $\mathrm{Fil}\ M$, and a φ-linear map $\varphi_M : M \to M$, satisfying the following conditions :

(1) $\varphi_M(\mathrm{Fil}\ M) \subset I.M$.
(2) $\mathcal{N}^{\geq 1}\Delta_R.M \subset \mathrm{Fil}\ M$ and $M/\mathrm{Fil}\ M$ is a finite locally free R-module.
(3) $\varphi_M(\mathrm{Fil}\ M)$ generates $I.M$ as a Δ_R-module.

For a prismatic Dieudonné module (M, φ_M) the linearization of the morphism $\varphi_M\colon M \to M$ is an isomorphism after inverting a generator $\tilde{\xi}$ of I and in particular is injective, since $\varphi^* M$ is $\tilde{\xi}$-torsion free.

The last condition in Definition 4.1.10 implies in particular that if $(M, \mathrm{Fil}\ M, \varphi_M)$ is a filtered prismatic Dieudonné module over R, then (M, φ_M) is a prismatic Dieudonné module over R.

Remark 4.1.11. If R is perfectoid, one has

$$(\Delta_R, I) = (A_{\mathrm{inf}}(R), (\tilde{\xi})).$$

A prismatic Dieudonné module is the same thing as a *minuscule Breuil-Kisin-Fargues module* ([10]) over $A_{\mathrm{inf}}(R)$ with respect to $\tilde{\xi}$.

Remark 4.1.12. Assume that R is quasi-regular semiperfect, i.e. R is quasi-regular semiperfectoid and $pR = 0$. Let (M, φ_M) be a prismatic Dieudonné module over R. Let $N \subset M/\mathcal{N}^{\geq 1}\Delta_R M$ be a locally direct summand, and define $\mathrm{Fil}\ M$ to be the inverse image of N in M. Then the collection $(M, \mathrm{Fil}\ M, \varphi_M)$ is a filtered prismatic Dieudonné module over R if and only if N is an "admissible" filtration in the sense of Grothendieck on the Dieudonné module (M, φ_M, V_M), where $V_M = \varphi_M^{-1}.p$ (which makes sense by the assumption that (M, φ_M) is a prismatic Dieudonné module and the p-torsion freeness of Δ_R). For a proof of this, see [16, Lemma 2.5.1]).

Proposition 4.1.13. *Let R be a quasi-regular semiperfectoid ring. The functor*

$$\mathcal{M} \mapsto v^*\mathcal{M}(\mathbb{\Delta}_R, I)$$

of evaluation on the initial prism $(\mathbb{\Delta}_R, I)$ induces an equivalence between the category of (filtered) prismatic Dieudonné crystals over R and the category of (filtered) prismatic Dieudonné modules over R.

Proof. Use Proposition 4.1.4, Proposition 4.1.2, the fact that

$$\mathbb{\Delta}_R = R\Gamma((R)_{\mathrm{qsyn}}, \mathcal{O}^{\mathrm{pris}}) \ ; \ R = R\Gamma((R)_{\mathrm{qsyn}}, \mathcal{O})$$

and that finite locally free $\mathcal{O}_{\mathbb{\Delta}}$-modules (resp. finite locally free $\mathcal{O}$-modules) are equivalent to finite locally free $\mathbb{\Delta}_R$-modules (resp. finite locally free R-modules). $\square$

Definition 4.1.14. We denote by $\mathrm{DM}(R)$ the category of prismatic Dieudonné modules over R (with morphisms commuting with the Frobenius) and by $\mathrm{DF}(R)$ the category of filtered prismatic Dieudonné modules over R (with morphisms commuting with the Frobenius and respecting the filtration).

Proposition 4.1.13 shows that the possible conflict of notation is not an issue : for R quasi-regular semiperfectoid, the two categories denoted by $\mathrm{DM}(R)$ are naturally equivalent, and similarly for $\mathrm{DF}(R)$.

The forgetful functor

$$(M, \mathrm{Fil}\ M, \varphi_M) \mapsto (M, \varphi_M)$$

from $\mathrm{DF}(R)$ to $\mathrm{DM}(R)$ is faithful. It is not essentially surjective, nor (a priori) fully faithful in general. But it is an equivalence for one important class of quasi-syntomic rings.

Lemma 4.1.15. *Let R be a perfectoid ring. Then the forgetful functor*

$$\mathrm{DF}(R) \to \mathrm{DM}(R)$$

is an equivalence. In fact, for $(M, \mathrm{Fil}M, \varphi_M) \in \mathrm{DF}(R)$ necessarily $\mathrm{Fil}M = \varphi_M^{-1}(IM)$.

Proof. If R is perfectoid, one has

$$\mathbb{\Delta}_R = A_{\mathrm{inf}}(R) \ ; \ \mathcal{N}^{\geq 1}\mathbb{\Delta}_R = \xi A_{\mathrm{inf}}(R),$$

where $\tilde{\xi} := \varphi(\xi)$ is a generator of I.

The argument of [16, Lemma 2.1.16] shows that the functor

$$(M, \mathrm{Fil}\ M, \varphi_M) \mapsto (\mathrm{Fil}\ M, \frac{\xi}{\tilde{\xi}}\varphi_M)$$

induces an equivalence betwwen $\mathrm{DF}(R)$ and $\mathrm{DM}(R')$, with $R' = A_{\mathrm{inf}}(R)/\xi$ (the key point is that $\mathcal{N}^{\geq 1}\mathbb{\Delta}_R$ is principal, so that Fil M is a projective $\mathbb{\Delta}_R$-module ; in particular, the linearization of the divided Frobenius $\varphi_M/\tilde{\xi}$ identifies $\varphi^*\mathrm{Fil}\ M$ and M). As φ is bijective on $\mathbb{\Delta}_R = A_{\mathrm{inf}}(R)$, base change along φ is also an equivalence between $\mathrm{DM}(R')$ and $\mathrm{DM}(R)$. The composite functor sends $\underline{M} \in \mathrm{DF}(R)$ to $(M, \varphi_M) \in \mathrm{DM}(R)$. $\square$

The definition of filtered prismatic Dieudonné crystals is inspired by classical definitions in crystalline Dieudonné theory, which are themselves generalized and abstracted in the theory of frames and windows. To end this section, let us shortly recall the general notions of *frame* and *window*, and the connection with the definitions above.

Definition 4.1.16. A *frame* $\underline{A} = (A, \mathrm{Fil}\ A, \varphi, \varphi_1)$ consists of (classically) (p,d)-adically complete rings A and $R = A/\mathrm{Fil}\ A$, for some $d \in A$ and some ideal Fil A, a lift of Frobenius φ, a φ-linear map $\varphi_1 : \mathrm{Fil}\ A \to A$ such that $\varphi = \varpi\varphi_1$ on Fil A, with $\varpi = \varphi(d)$.

Remark 4.1.17. In many situations (such as those considered in this paper), the image of φ_1 will always generate the unit ideal of A.

Here is an important source of examples.

Example 4.1.18. Let $(A, I = (d))$ be an oriented prism. There are two natural ways of attaching a frame to $(A,(d))$. One possibility is to consider the frame

$$\underline{A}_d = (A, (d), \varphi, \varphi_1),$$

where φ_1 is defined by $\varphi_1(dx) = \varphi(x)$ (recall that A is d-torsion free). The other possibility is to consider the frame

$$\underline{A}_{\mathrm{Nyg}} = (A, \mathcal{N}^{\geq 1}A, \varphi, \varphi_1)$$

where $\varphi_1 := \varphi/d$ on $\mathcal{N}^{\geq 1}A$ (using again that A is d-torsion free). Note that in the first case, the divided Frobenius is with respect to $\varphi(d)$, whereas in the second case the divided Frobenius is with respect to d.

Definition 4.1.19. A *window* $\underline{M} = (M, \mathrm{Fil}\ M, \varphi_M, \varphi_{M,1})$ over a frame $\underline{A}$ consists of a finite locally free A-module M, an A-submodule Fil $M \subset M$, and φ-linear maps $\varphi_M : M \to M$ and $\varphi_{M,1} : \mathrm{Fil}\ M \to M$, such that :

- Fil $A \cdot M \subset$ Fil M and $M/\mathrm{Fil}\ M$ is a finite locally free R-module.
- If $a \in \mathrm{Fil}\ A$, $m \in M$, $\varphi_{M,1}(am) = \varphi_1(a)\varphi_M(m)$.
- If $m \in \mathrm{Fil}\ M$, $\varphi_M(m) = \varpi\varphi_{M,1}(m)$.
- $\varphi_{M,1}(\mathrm{Fil}\ M) + \varphi_M(M)$ generates M as an A-module.

A morphism of windows is an A-linear map preserving the filtrations and commuting with φ_M and $\varphi_{M,1}$. The category of windows over $\underline{A}$ is denoted by $\mathrm{Win}(\underline{A})$.

Remark 4.1.20. If the surjectivity condition on the image of φ_1 of Remark 4.1.17 is satisfied, then the third point of the previous definition follows from the second and the last one simply says that $\varphi_{M,1}(\mathrm{Fil}\ M)$ generates M.

Example 4.1.21. Let $(A,(d))$ is an oriented prism. The category of windows over the frame $\underline{A}_d$ of Example 4.1.18 is equivalent to the category $\mathrm{BK}(A)$ of *minuscule Breuil-Kisin modules over* A, that is, to the category of pairs (M, φ_M) where M is a finite projective A-module and $\varphi_M : M \to M$ a φ-linear map such that its linearization has cokernel killed by d and projective as an A/d-module. See [16, Lemma 2.1.16] for a proof. The functor sends a window $(M, \mathrm{Fil}M, \varphi_M, \varphi_{M,1})$ to $(\mathrm{Fil}M, d.\varphi_{M,1})$.

The other frame structure attached to an oriented prism discussed in Example 4.1.18 is the one which is connected to filtered prismatic Dieudonné modules. More precisely, let R be a quasi-regular semiperfectoid ring, with initial prism (Δ_R, I). If one chooses a generator $\tilde{\xi}$ of I, one gets, for each filtered Dieudonné module $(M, \mathrm{Fil}\ M, \varphi_M)$ over R, a divided Frobenius $\varphi_{M,1}$ on Fil M, by dividing by $\tilde{\xi}$ (recall that Δ_R is $\tilde{\xi}$-torsion free by the Hodge-Tate comparison). Then :

Proposition 4.1.22. *Let R be a quasi-regular semiperfectoid ring. After the choice of a generator $\tilde{\xi}$ of the ideal I of the prism $(\mathbb{A}_R, I)$, the category* $\mathrm{DF}(R)$ *is equivalent to the category* $\mathrm{Win}(\underline{\mathbb{A}}_R)$*, where $\underline{\mathbb{A}}_{R,\mathrm{Nyg}}$ is the frame associated to $(\mathbb{A}_R, I)$ and $\tilde{\xi}$, as in Example 4.1.18.*

Proof. Since $\mathbb{A}_R$ is $\tilde{\xi}$-torsion free, the divided Frobenius on the filtration is determined by φ_M. □

Therefore, the theory of filtered prismatic Dieudonné modules fits into the general formalism of windows[21]. For example, filtered prismatic Dieudonné modules have *normal decompositions*, as we now explain. This simple fact will be very helpful later when describing the essential image of the filtered prismatic Dieudonné functor.

Let us first recall some facts about henselian pairs. Let A be a ring and let $I \subseteq A$ be an ideal. We recall that the pair (A, I) is henselian if I is contained in the Jacobson radical of A and if for any monic polynomial $f \in A[T]$ and each factoriztion $\overline{f} = g_0 h_0$ with $g_0, h_0 \in A/I[T]$ monic and generating the unit ideal, there exists a factorization $f = gh$ with g, h monic and $g_0 = \overline{g}$, $h_0 = \overline{h}$ (cf. [49, Tag 09XE]).

If I is locally nilpotent[22] or A is I-adically complete, then the pair (A, I) is henselian (cf. [49, Tag 0ALI], [49, Tag 0ALJ]).

For us the following well-known property of henselian pairs will be important (cf. [18, Lemma 4.20]).

Lemma 4.1.23. *Let (A, I) be an henselian pair. The base change $M \mapsto M \otimes_A A/I$ induces a bijection on isomorphism classes of finite projective modules over A, resp. A/I.*

Proof. If M, N are finite projective A-modules, then any isomorphism $M/IM \cong N/IN$ can be lifted to a morphism $M \to N$ by projectivity of M. As $I \subseteq A$ lies in the Jacobson radical of A this lifted homomorphism is then automatically an isomorphism. Moreover, any finite projective A/I-module can be lifted to a finite projective A-module by [49, Tag 0D4A]. □

Now, we provide the proof that $\mathbb{A}_R$ is henselian along $\mathcal{N}^{\geq 1}\mathbb{A}_R = \ker(\theta \colon \mathbb{A}_R \to R)$. We learned the argument from [37, Remark 5.2].

Lemma 4.1.24. *The pair $(\mathbb{A}_R, \ker(\theta))$ is henselian.*

Proof. Because $\mathbb{A}_R$ is (p, ξ)-adically complete it suffices to prove that the pair

$$(\mathbb{A}_R/(p, \xi), (p, \ker(\theta))/(p, \xi))$$

is henselian (cf. [49, Tag 0DYD]). We know $\ker(\theta) = \mathcal{N}^{\geq 1}\mathbb{A}_R$. Hence, for every element $x \in \ker(\theta)$, $x^p \in (p, \tilde{\xi})$. As locally nilpotent ideals are henselian the claim follows. □

[21]Nevertheless, as is visible in the proposition, we are in a situation where the subtle aspects of the theory of windows, which have to do with the divided Frobenius in presence of torsion, do not show up. They should if one tries to set up the theory for p-complete rings which are not necessarily quasi-syntomic. See [37] for characteristic p rings.

[22]That is, every element in I is nilpotent.

Proposition 4.1.25. *Let $(M, \mathrm{Fil}\ M, \varphi_M)$ be a filtered prismatic Dieudonné module over R. Then there exist finite projective Δ_R-modules L, T such that $M = L \oplus T$ and* $\mathrm{Fil}\ M = L \oplus \mathcal{N}^{\geq 1}\Delta_R T$. *Moreover, given L, T there exists a bijection between isomorphisms $\psi\colon \varphi^*(L \oplus T) \to L \oplus T$ and semi-linear endomorphisms φ'_M such that $(M, \mathrm{Fil}\ M, \varphi'_M)$ is a filtered prismatic Dieudonné module.*

Proof. This follows from the fact that Δ_R is henselian along $\mathcal{N}^{\geq 1}\Delta_R$ (Lemma 4.1.24) and Lemma 4.1.23. Namely, the module $R \otimes_{\Delta_R} M$ decomposes, as $M/\mathrm{Fil}\ M$ is finite projective, into a direct sum $R \otimes_{\Delta_R} M \cong M/\mathrm{Fil}\ M \oplus Q$ for some finite projective R-module Q. Let L, T be finite projective Δ_R-modules such that L is a lift of Q and T a lift of $M/\mathrm{Fil}\ M$. We can then lift the decomposition $R \otimes_{\Delta_R} M$ to a decomposition $M = L \oplus T$ by projectivity. The property $\mathrm{Fil}\ M = L \oplus \mathcal{N}^{\geq 1}\Delta_R T$ follows. The last claim is [33, Lemma 2.5]. $\square$

We record some statements which are later used to prove essential surjectivity for the filtered prismatic Dieudonné functor.

For a ring A with an endomorphism $\varphi\colon A \to A$ we denote by $\varphi - \mathrm{Mod}_A$ the category of φ-modules over A, i.e., the category of pairs (M, φ_M) with M a finite projective A-module and $\varphi_M\colon \varphi^* M \cong M$ an isomorphism.

Lemma 4.1.26. *Let $A \to B$ be a surjection of prisms with kernel $J \subseteq A$. Assume that the Frobenius φ of A is topologically nilpotent on J and that (A, J) is henselian. Then the functor*

$$\varphi - \mathrm{Mod}_A \to \varphi - \mathrm{Mod}_B,\ (M, \varphi_M) \mapsto (M \otimes_A B, \varphi_M \otimes_A B)$$

is an equivalence.

Proof. To prove fully faithfulness it suffices to show (by passing to internal hom's) that for every φ-module (M, φ_M) over A the map

$$M^{\varphi_M = 1} \to (M/JM)^{\varphi_M = 1}$$

is bijective. Let $m \in M^{\varphi_M = 1} \cap JM$ and write $m = \sum\limits_{i=1}^{n} a_i m_i$ with $a_i \in J$ and $m_i \in M$. Then

$$m = \varphi_M^j(m) = \sum_{i=1}^{n} \varphi^j(a_i)\varphi_M^j(m_i)$$

where the $\varphi^j(a_i)$ converge to 0 if $j \to \infty$ by our assumption on φ. Thus $m = \varphi_M^j(m) \to 0$ if $j \to \infty$ and therefore $m = 0$, which proves injectivity. Conversely, let $m \in M$ and assume that $\varphi_M(m) \equiv m$ modulo JM. Write

$$z := \varphi_M(m) - m \in JM.$$

As above the sequence $\varphi_M^j(z)$ converges to 0 if $j \to \infty$. Set

$$\tilde{m} := m + \sum_{j=0}^{\infty} \varphi_M^j(z).$$

Then $\tilde{m} \equiv m$ modulo JM and $\varphi_M(\tilde{m}) = \tilde{m}$. Thus we showed that

$$M^{\varphi_M = 1} \cong (M/JM)^{\varphi_M = 1}$$

and the functor $\varphi - \mathrm{Mod}_A \to \varphi - \mathrm{Mod}_B$ is fully faithful and we are left with essential surjectivity. For this let $(N, \varphi_N) \in \varphi - \mathrm{Mod}_B$. By assumption A is henselian along J and thus we can write $N \cong M \otimes_A B$ for some finite projective A-module M. Using projectiviy of $\varphi^* M$ over A we can lift $\varphi_N \colon \varphi^* N \to N$ to some homomorphism $\varphi_M \colon \varphi^* M \to M$. As J lies in the radical of A the homomorphism φ_M will automatically be an isomorphism as φ_N is. Thus, we have lifted (N, φ_N) to (M, φ_M), which finishes the proof. □

The following statement is similar to [33, Lemma 2.12] or [30, Appendix A.4].

Lemma 4.1.27. *Let $(A, (\tilde{\xi})) \to (B, (\tilde{\xi}))$ be a surjection of oriented prisms with kernel J contained in $\mathcal{N}^{\geq 1}A$. Assume that φ_1 is topologically nilpotent on J and that (A, J) is henselian. Then the base change functor induces an equivalence :*

$$\mathrm{Win}(\underline{A}) \simeq \mathrm{Win}((B, \mathcal{N}^{\geq 1}A/J, A/\mathcal{N}^{\geq 1}A, \varphi, \varphi_1)).$$

We note that $\varphi_1(J) \subseteq J$ as B is $\tilde{\xi}$-torsion free and $\varphi(j) = \tilde{\xi}\varphi_1(j)$ in A. Thus the condition that φ_1 is topologically nilpotent on J makes sense. Moreover, $\varphi_1(J) \subseteq J$ implies that $(B, \mathcal{N}^{\geq 1}A/J, A/\mathcal{N}^{\geq 1}A, \varphi, \varphi_1)$ is indeed a well-defined frame.

Proof. By the existence of normal decompositions and the fact that A is henselian along J the base change functor

$$\mathrm{Win}(\underline{A}) \to \mathrm{Win}(\underline{B})$$

is essentially surjective. Let $\underline{M}, \underline{N}$ be two windows over $\underline{A}$. We want to prove that

$$\mathrm{Hom}_{\underline{A}}(\underline{M}, \underline{N}) \cong \mathrm{Hom}_{\underline{B}}(\underline{M}/J, \underline{N}/J)$$

where $\underline{M}/J, \underline{N}/J$ denote the base change of $\underline{M}, \underline{N}$ to $\underline{B}$. The idea of proof is similar to Lemma 4.1.26 (and [33, Theorem 3.2]). Let

$$\beta \colon M \to JN$$

be an arbitrary homomorphism of A-modules. Then the A-module homomorphism

$$U(\beta) \colon M \to JN,\ m \mapsto 1/\tilde{\xi}\varphi_N(\mathrm{Id} \otimes \beta)(\varphi_M^{-1}(\tilde{\xi}m))$$

is well-defined. Indeed, $\varphi_M \colon \varphi^* M \to M$ is injective with cokernel killed by $\tilde{\xi}$ (which follows from the fact that $\varphi_{M,1}(\mathrm{Fil}M)$ generates M and that $M, \varphi^*(M)$ are $\tilde{\xi}$-torsion free) and thus on $\tilde{\xi}M$ there exists a partial inverse $\varphi_M^{-1} \colon \tilde{\xi}M \to \varphi^* M$ of φ_M. Moreover, as β has image in JN the composition $\varphi_N(\mathrm{Id} \otimes \beta)$ has image in $\tilde{\xi}N$. By our assumption on topologically nilpotence of φ_1 on J the endomorphism $\varphi_{N,1} \colon JN \to JN$ is topologically nilpotent. Hence, for every $\beta \colon M \to JN$ the sequence

$$\beta, U(\beta), U(U(\beta)), \ldots, U^n(\beta), \ldots$$

converges to 0. Now let $\alpha \colon M \to N$ be a homomorphism of windows such that $\alpha \equiv 0$ modulo J. Then $U^n(\alpha) = \alpha$ for all n because $\alpha \circ \varphi_M = \varphi_N \circ \alpha$, which implies $\alpha = 0$ as the sequence $U^n(\alpha)$ converges to 0 as we saw above. Conversely, assume that $\alpha \colon M \to N$ is an A-module homomorphism, such that α modulo J is an homomorphism of windows over $\underline{B}$. Then α maps $\mathrm{Fil}M$ to $\mathrm{Fil}N$ because this can be checked modulo J. Set

$$\beta := U(\alpha) - \alpha \colon M \to 1/\tilde{\xi}N.$$

Then $\beta(M) \subseteq JN$. Therefore the homomorphism

$$\tilde{\alpha} \colon M \to N,\ m \mapsto \alpha(m) + \sum_{n=0}^{\infty} U^n(\beta)(m)$$

is well-defined. Moreover, $\alpha \equiv \tilde{\alpha}$ modulo J and $\tilde{\alpha}$ is a homomorphism of windows over $\underline{A}$. $\square$

From the proof of the last lemma, one can also extract the following statement.

Lemma 4.1.28. *Let $R \to R'$ be a morphism of quasi-regular semiperfectoid rings such that $J = \ker(\mathbb{\Delta}_R \to \mathbb{\Delta}_{R'})$ is contained in $\mathcal{N}^{\geq 1}\mathbb{\Delta}_R$, stable by φ_1 and such that φ_1 is topologically nilpotent on J (for some, or equivalently any, choice of a generator of the idealI defining the prism structure of $\mathbb{\Delta}_R$). Then the base change functors*

$$\mathrm{DM}(R) \to \mathrm{DM}(R') \quad ; \quad \mathrm{DF}(R) \to \mathrm{DF}(R')$$

are faithful.

Proof. It is enough to prove that the first functor is faithful. For this, one uses the exact same argument used in the proof of Lemma 4.1.27. $\square$

Remark 4.1.29. More generally, if one has a morphism of frames $\underline{A} \to \underline{A}'$, whose kernel J is contained in Fil A, stable by φ_1, and such that φ_1 is topologically nilpotent on J, the same proof shows that the base change functor

$$\mathrm{Win}(\underline{A}) \to \mathrm{Win}(\underline{A}')$$

is faithful.

4.2. **Definition of the filtered prismatic Dieudonné functor.** In this subsection we define the filtered prismatic Dieudonné crystals of p-divisible groups over quasi-syntomic rings and prove some formal properties of them. More difficult properties, like the crystal property or local freeness, will be proved later (cf. Section 4.6) after discussing the case of abelian schemes first (cf. Section 4.5).

Let $R \in \mathrm{QSyn}$ be a quasi-syntomic ring and let $(R)_{\mathbb{\Delta}}$ be its absolute prismatic site. We recall from Proposition 4.1.4 that the category of finite locally free crystals on $(R)_{\mathbb{\Delta}}$ is equivalent to the category of finite locally free $\mathcal{O}^{\mathrm{pris}}$-modules on the small quasi-syntomic site $(R)_{\mathrm{qsyn}}$ of R endowed with the quasi-syntomic topology.

Recall as well that there is an exact sequence

$$0 \to \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}} \to \mathcal{O}^{\mathrm{pris}} \to \mathcal{O} \to 0$$

where $\mathcal{O}$ is the structure sheaf $S \in (R)_{\mathrm{qsyn}} \mapsto S$ on $(R)_{\mathrm{qsyn}}$ (cf. Proposition 4.1.2).

Definition 4.2.1. Let G be a p-divisible group over R. We define

$$\mathcal{M}_{\mathbb{\Delta}}(G) := \mathcal{E}xt^1_{(R)_{\mathrm{qsyn}}}(G, \mathcal{O}^{\mathrm{pris}})$$
$$\mathrm{Fil}\mathcal{M}_{\mathbb{\Delta}}(G) := \mathcal{E}xt^1_{(R)_{\mathrm{qsyn}}}(G, \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}})$$

and $\varphi_{\mathcal{M}_{\mathbb{\Delta}}(G)}$ as the endomorphism of $\mathcal{M}_{\mathbb{\Delta}}(G)$ induced from the endomorphism φ on $\mathcal{O}^{\mathrm{pris}}$. We call $(\mathcal{M}_{\mathbb{\Delta}}(G), \varphi_{\mathcal{M}_{\mathbb{\Delta}}(G)})$ the *prismatic Dieudonné crystal* of G and the data $\underline{\mathcal{M}}_{\mathbb{\Delta}}(G) := (\mathcal{M}_{\mathbb{\Delta}}(G), \mathrm{Fil}\mathcal{M}_{\mathbb{\Delta}}G, \varphi_{\mathcal{M}_{\mathbb{\Delta}}(G)})$ the *filtered prismatic Dieudonné crystal* of G.

Remark 4.2.2. Beware that the prismatic Dieudonné crystal of a p-divisible group is a sheaf on the quasi-syntomic site, not on the prismatic site. In particular, it is not a crystal on the prismatic site of R, but rather the push-forward along v of a crystal on the prismatic site (as will be proved later). We hope that this choice of terminology does not create too much confusion ; from the mathematical point of view, it is justified by Proposition 4.1.4.

Fix a p-divisible group G over R. We check some easy properties of $\underline{\mathcal{M}_{\Delta}}(G)$. Recall that there is the natural prismatic Cartier divisor $\mathcal{I}^{\mathrm{pris}} \subseteq \mathcal{O}^{\mathrm{pris}}$.

Lemma 4.2.3. *The morphism* $\mathrm{Fil}\mathcal{M}_{\Delta}(G) \to \mathcal{M}_{\Delta}(G)$ *is injective and*

$$\varphi_{\mathcal{M}_{\Delta}(G)}(\mathrm{Fil}\mathcal{M}_{\Delta}(G)) \subseteq \mathcal{I}^{\mathrm{pris}}\mathcal{M}_{\Delta}(G).$$

Proof. The injectivity follows from $\mathcal{H}om(G, \mathcal{O}) = 0$ as G is p-divisible and $\mathcal{O}$ is a p-complete sheaf. For the second statement we claim that

$$\mathcal{I}^{\mathrm{pris}}\mathcal{M}_{\Delta}(G) \cong \mathcal{E}xt^1_{(R)_{\mathrm{qsyn}}}(G, \mathcal{I}^{\mathrm{pris}}).$$

For this it suffices to see that $\mathcal{H}om(G, \mathcal{O}^{\mathrm{pris}}/\mathcal{I}^{\mathrm{pris}}) = 0$. But $\mathcal{H}om(G, \mathcal{O}^{\mathrm{pris}}/\mathcal{I}^{\mathrm{pris}})$ embeds into $\mathcal{H}om(G, v_*(\overline{\mathcal{O}}_{\Delta}))$ and this sheaf is zero as $v_*(\overline{\mathcal{O}}_{\Delta})$ is p-complete and G p-divisible. As $\varphi(\mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}) \subseteq \mathcal{I}^{\mathrm{pris}}$ the map $\varphi_{\mathcal{M}_{\Delta}(G)}$ will thus send $\mathrm{Fil}\mathcal{M}_{\Delta}(G)$ into $\mathcal{I}^{\mathrm{pris}}\mathcal{M}_{\Delta}(G)$. $\square$

In [5], the crystalline Dieudonné crystal of a p-divisible group is defined via the sheaf of local extensions on the crystalline site. There is a similar description of the filtered prismatic Dieudonné crystal. Let

$$u\colon \mathrm{Shv}(R)_{\Delta} \to \mathrm{Shv}(R)_{\mathrm{QSYN}}$$

be the morphism from the prismatic to the big quasi-syntomic topos constructed in Corollary 3.3.10. If $\mathcal{F}$ is a sheaf on $(R)_{\mathrm{QSYN}}$, $u^{-1}\mathcal{F}$ is simply the sheafification of the functor sending a prism $(A, I) \in (R)_{\Delta}$ to $\mathcal{F}(A/I)$.

Let $\mathrm{Shv}(R)_{\mathrm{qsyn}}$ be the small pro-syntomic topos of R and

$$\varepsilon\colon \mathrm{Shv}(R)_{\mathrm{QSYN}} \to \mathrm{Shv}(R)_{\mathrm{qsyn}}$$

be the natural projection. Then

$$v = \varepsilon \circ u\colon \mathrm{Shv}(R)_{\Delta} \to \mathrm{Shv}(R)_{\mathrm{qsyn}}$$

and thus

$$v^{-1} = u^{-1} \circ \varepsilon^{-1}.$$

Lemma 4.2.4. *There are canonical isomorphisms*

$$\mathcal{M}_{\Delta}(G) \cong v_*(\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}(G), \mathcal{O}_{\Delta}))$$
$$\mathrm{Fil}\mathcal{M}_{\Delta}(G) \cong v_*(\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}(G), \mathcal{N}^{\geq 1}\mathcal{O}_{\Delta})).$$

Proof. By adjunction there is a canonical isomorphism

$$R\mathcal{H}om(G, Rv_*(\mathcal{O}_{\Delta})) \cong Rv_*(R\mathcal{H}om(v^{-1}G, \mathcal{O}_{\Delta})).$$

The p-divisible group G is the colimit of the p^n-torsion $G[p^\infty]$ on the small site $(R)_{\mathrm{qsyn}}$. As $\varepsilon^{-1}(G[p^n]) = G[p^n]$, or more precisely $\varepsilon^{-1}(G[p^n])$ is the sheaf on the big quasi-syntomic site represented by $G[p^n]$, one can conclude $\varepsilon^{-1}(G) = G$ by passing to the colimit. In particular, $v^{-1}(G) = u^{-1}(G)$. Thus we obtain a canonical isomorphism

$$R\mathcal{H}om(G, Rv_*(\mathcal{O}_{\Delta})) \cong Rv_*(R\mathcal{H}om(u^{-1}(G), \mathcal{O}_{\Delta})).$$

It suffices to see that $\mathcal{M}_{\Delta}(G)$, resp. $v_*(\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}(G), \mathcal{O}_{\Delta}))$, are the first cohomology sheaves on both sides (and similarly with $\mathcal{O}_{\Delta}$ replaced by $\mathcal{N}^{\geq 1}\mathcal{O}_{\Delta}$). The sheaves

$$\mathcal{H}om(G, R^1v_*(\mathcal{O}_{\Delta})), \mathcal{H}om(u^{-1}(G), \mathcal{O}_{\Delta})$$

are 0 as G is p-divisible and the target p-complete (similarly for $\mathcal{O}_{\Delta}$ replaced by $\mathcal{N}^{\geq 1}\mathcal{O}_{\Delta}$). This implies the statement. $\square$

Using the p-adic Tate module T_pG of G, i.e., the inverse limit

$$\varprojlim_n G[p^n]$$

of sheaves on $(R)_{\mathrm{qsyn}}$, one can give a more explicit description of the prismatic Dieudonné crystal $\mathcal{M}_{\Delta}(G)$.

Lemma 4.2.5. *Define the universal cover* $\tilde{G} := \varprojlim_p G$ *of* G. *Then the sequences*

$$0 \to T_pG \to \tilde{G} \to G \to 0$$
$$0 \to u^{-1}T_pG \to u^{-1}\tilde{G} \to u^{-1}G \to 0$$

of sheaves on $(R)_{\mathrm{qsyn}}$ *resp.* $(R)_{\Delta}$ *are exact for the quasi-syntomic topology.*

Proof. Exactness of the second follows from exactness of the first and exactness of u^{-1} (cf. Corollary 3.3.11). The sequence

$$0 \to G[p^n] \to G \to G \to 0$$

is exact for the quasi-syntomic topology as each $G[p^n]$ is syntomic over R. Then exactness of $0 \to T_pG \to \tilde{G} \to G \to 0$ follows by passing to the limit and using repleteness (in the sense of [13, Section 3]) of $(R)_{\mathrm{qsyn}}$. $\square$

The following lemma will be useful when describing the filtered prismatic Dieudonné crystals of $\mathbb{Q}_p/\mathbb{Z}_p$ and μ_{p^∞}.

Lemma 4.2.6. *There are a canonical isomorphisms*

$$\mathcal{M}_{\Delta}(G) \cong \mathcal{H}om_{(R)_{\mathrm{qsyn}}}(T_pG, \mathcal{O}^{\mathrm{pris}}) \cong v_*\mathcal{H}om_{(R)_{\Delta}}(u^{-1}(T_pG), \mathcal{O}_{\Delta})$$

and similarly for $\mathrm{Fil}\mathcal{M}_{\Delta}(G)$ *and* $\mathcal{N}^{\geq 1}\mathcal{O}_{\Delta}, \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}$.

Proof. This follows from Lemma 4.2.5 and the fact that

$$R\mathcal{H}om_{(R)_{\Delta}}(u^{-1}(\tilde{G}), \mathcal{O}_{\Delta}) = 0 \quad ; \quad R\mathcal{H}om_{(R)_{\mathrm{qsyn}}}(\tilde{G}, \mathcal{O}^{\mathrm{pris}}) = 0$$

as $\mathcal{O}_{\Delta}, \mathcal{O}^{\mathrm{pris}}$ are derived p-complete sheaves and $\tilde{G}$ is a $\mathbb{Q}_p$-vector space. The same argument works for $\mathrm{Fil}\mathcal{M}_{\Delta}(G)$ as well. $\square$

Remark 4.2.7. The universal vector extension $E(G)$ of G can be seen as an extension of sheaves on $(R)_{\mathrm{qsyn}}$:

$$0 \to \omega_{\check{G}} \to E(G) \to G \to 0.$$

It is defined as in [42] (this makes sense since R is p-complete), or equivalently as the push-out of the universal cover exact sequence

$$0 \to T_pG \to \tilde{G} \to G \to 0$$

along the *Hodge-Tate map*

$$HT : T_pG \to \omega_{\check{G}},$$

which sends $f \in T_pG = \mathrm{Hom}_R(\mathbb{Q}_p/\mathbb{Z}_p, G)$, viewed by Cartier duality as an element of $\mathrm{Hom}_R(\mu_{p^\infty}, \check{G})$, to f^*dT/T, dT/T being the canonical generator of $\omega_{\mu_{p^\infty}}$. Is there a way to use Lemma 4.2.6 to relate the prismatic Dieudonné module to the dual of the Lie algebra of $E(G)$?

Assume now that R is quasi-regular semiperfectoid. Then, by Proposition 4.1.4, the category of finite locally free crystals on $(R)_{\Delta}$ is equivalent to the category of finite projective Δ_R-modules by evaluating a crystal on the initial prism Δ_R. Similarly, finite locally free $\mathcal{O}^{\mathrm{pris}}$-modules on $(R)_{\mathrm{qsyn}}$ are equivalent to finite projective Δ_R by evaluating a finite locally free $\mathcal{O}^{\mathrm{pris}}$-module $\mathcal{M}$ on R. This allows the following simplification of the definition of the filtered prismatic Dieudonné crystal of a p-divisible group G over R.

Definition 4.2.8. Let R be quasi-regular semiperfectoid and let G be a p-divisible group over R. Define

$$M_{\Delta}(G) := \mathrm{Ext}^1_{(R)_{\mathrm{qsyn}}}(G, \mathcal{O}^{\mathrm{pris}}) \cong \mathrm{Ext}^1_{(R)_{\Delta}}(u^{-1}(G), \mathcal{O}_{\Delta})$$
$$\mathrm{Fil}M_{\Delta}(G) := \mathrm{Ext}^1_{(R)_{\mathrm{qsyn}}}(G, \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}) \cong \mathrm{Ext}^1_{(R)_{\Delta}}(u^{-1}(G), \mathcal{N}^{\geq 1}\mathcal{O}_{\Delta})$$

and $\varphi_{M_{\Delta}(G)}$ as the endomorphism induced by φ on $\mathcal{O}^{\mathrm{pris}}$. We call

$$(M_{\Delta}(G), \varphi_{M_{\Delta}(G)})$$

the *prismatic Dieudonné module* of G and

$$\underline{M_{\Delta}}(G) := (M_{\Delta}(G), \mathrm{Fil}M_{\Delta}(G), \varphi_{M_{\Delta}(G)})$$

the *filtered prismatic Dieudonné module* of G.

We will see later that $\underline{M_{\Delta}}(G)$ is indeed a filtered prismatic Dieudonné module in the sense of Definition 4.1.9. Moreover, $\underline{M_{\Delta}}(G)$ is the evaluation of the filtered prismatic Dieudonné crystal $\underline{\mathcal{M}_{\Delta}}(G)$ as follows from the local-global spectral sequence

$$E_2^{ij} = H^i(\mathrm{Spf}(R), \mathcal{E}xt^j_{(R)_{\mathrm{qsyn}}}(G, \mathcal{O}^{\mathrm{pris}})) \Rightarrow \mathrm{Ext}^{i+j}_{(R)_{\mathrm{qsyn}}}(G, \mathcal{O}^{\mathrm{pris}})$$

by the vanishing of the sheaf $\mathcal{H}om_{(R)_{\mathrm{qsyn}}}(G, \mathcal{O}^{\mathrm{pris}})$. Thus under the equivalence from Proposition 4.1.13 the filtered prismatic Dieudonné crystal $\underline{\mathcal{M}_{\Delta}}(G)$ corresponds to the filtered prismatic Dieudonné module $\underline{M_{\Delta}}(G)$.

4.3. **Comparison with former constructions.** In this section we prove a comparison of the filtered prismatic Dieudonné functor $\underline{\mathcal{M}_{\Delta}}$ with former constructions, in two special cases :

(1) For quasi-syntomic rings such that $pR = 0$, we relate $\mathcal{M}_{\Delta}$ to the crystalline Dieudonné functor of Berthelot-Breen-Messing [5], and more generally $\underline{\mathcal{M}_{\Delta}}$ to the functor considered by Lau in [37].
(2) For perfectoid rings, we relate the prismatic Dieudonné functor to the functor introduced by Scholze-Weinstein in [47, Appendix to Lecture XVII].

The intersection of these two cases is the case of perfect rings, which was historically the first to be studied. The situation for perfect fields is briefly discussed at the end of this section.

We start with the case of quasi-syntomic rings R with $pR = 0$. We want to compare the prismatic Dieudonné functor to the crystalline Dieudonné functor

$$G \mapsto \mathcal{E}xt^1_{(R/\mathbb{Z}_p)_{\mathrm{crys,pr}}}(i_*^{\mathrm{crys}}(G), \mathcal{O}_{\mathrm{crys}})$$

of [5]. Here $(R/\mathbb{Z}_p)_{\mathrm{crys,pr}}$ is the (big) crystalline site of R over $\mathbb{Z}_p$, $\mathcal{O}_{\mathrm{crys}}$ is the crystalline structure sheaf,

$$i^{\mathrm{crys}} \colon \mathrm{Shv}(R)_{\mathrm{pr}} \to \mathrm{Shv}(R/\mathbb{Z}_p)_{\mathrm{crys,pr}}$$

is the closed immersion (cf. [37, Lemma 8.1.]) and pr denotes the p-th root topology of [37, Definition 7.2.]. As in [37, Section 8] we define

$$\mathcal{O}^{\mathrm{crys}} := u_*^{\mathrm{crys}}(\mathcal{O}_{\mathrm{crys}})$$

as the pushforward of the crystalline structure sheaf $\mathcal{O}_{\mathrm{crys}}$ along the morphism

$$u^{\mathrm{crys}}\colon \mathrm{Shv}(R/\mathbb{Z}_p)_{\mathrm{crys,pr}} \to \mathrm{Shv}(R)_{\mathrm{pr}}$$

of topoi. Note that $i_*^{\mathrm{crys}} = (u^{\mathrm{crys}})^{-1}$, so we can rewrite the crystalline Dieudonné functor as

$$G \mapsto \mathcal{E}xt^1_{(R/\mathbb{Z}_p)_{\mathrm{crys,pr}}}((u^{\mathrm{crys}})^{-1}(G), \mathcal{O}_{\mathrm{crys}})$$

Let $\mathcal{J}^{\mathrm{crys}} \subseteq \mathcal{O}^{\mathrm{crys}}$ be the pushforward of the crystalline ideal sheaf $\mathcal{J}_{\mathrm{crys}} \subseteq \mathcal{O}_{\mathrm{crys}}$.

The following lemma is the basic input in the comparison of the prismatic and crystalline Dieudonné functor.

Lemma 4.3.1. *Let $R \to R'$ be morphism of characteristic p rings which is a p-th root morphism in the sense of* [37, Definition 7.2.]*, i.e., Zariski-locally (on* $\mathrm{Spec}(R')$ *and* $\mathrm{Spec}(R)$*) R' is obtained by sucessively adjoining p-th roots of some elements. Then there is a canonical isomorphism*

$$\mathcal{O}^{\mathrm{pris}}(R') \to \mathcal{O}^{\mathrm{crys}}(R')$$

identifying $\mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}(R')$ with $\mathcal{J}^{\mathrm{crys}}(R')$.

Note that R' is quasi-syntomic over R and thus $\mathcal{O}^{\mathrm{pris}}(R')$ is defined.

Proof. Using the sheaf property for the pr-topology we may assume that R' is semiperfect. Then R' is even quasi-regular semiperfect as it is quasi-syntomic over R. Hence,

$$\mathcal{O}^{\mathrm{pris}}(R') = \mathbb{A}_{R'} \cong A_{\mathrm{crys}}(R') = \mathcal{O}^{\mathrm{crys}}(R')$$

by Lemma 3.4.3. Moreover, the isomorphism in Lemma 3.4.3 identifies $\mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}(R')$ with $\mathcal{J}^{\mathrm{crys}}$. □

Let $(R)_{\mathrm{qsyn,pr}}$ be the category of quasi-syntomic R-algebras equipped with the pr-topology, and let

$$v^{\mathrm{crys}}\colon \mathrm{Shv}(R/\mathbb{Z}_p)_{\mathrm{crys,pr}} \to \mathrm{Shv}(R)_{\mathrm{qsyn,pr}}$$

be the morphism of topoi obtained by composing u^{crys} with restriction. Lemma 4.3.1 implies that the sheaves $\mathcal{O}^{\mathrm{pris}}$ and $\mathcal{O}^{\mathrm{crys}}$ on $(R)_{\mathrm{qsyn,pr}}$ are isomorphic. We note that the categories of finite locally free $\mathcal{O}^{\mathrm{crys}}$-modules on $(R)_{\mathrm{pr}}$ and finite locally free $\mathcal{O}^{\mathrm{crys}}_{|(R)_{\mathrm{qsyn,pr}}}$-modules on $(R)_{\mathrm{qsyn,pr}}$ are equivalent because for R quasi-regular semiperfect both categories identify with finite locally free $A_{\mathrm{crys}}(R)$-modules. Similarly, the category of filtered prismatic Dieudonné crystals over R and the category of filtered Dieudonné crystals of [37, Definition 8.11] are identified. These remarks give a meaning to the comparison contained in the next two results.

Theorem 4.3.2. *Let R be a quasi-syntomic ring with $pR = 0$ and G a p-divisible group over R. Then there is a canonical Frobenius equivariant isomorphism*

$$\mathcal{M}_{\mathbb{A}}(G) \cong v_*^{\mathrm{crys}}(\mathcal{E}xt^1_{(R/\mathbb{Z}_p)_{\mathrm{crys,pr}}}((u^{\mathrm{crys}})^{-1}(G), \mathcal{O}^{\mathrm{crys}}))$$

from the prismatic Dieudonné crystal of G (cf. Definition 4.2.1) to the push-forward of the crystalline Dieudonné crystal of G, which carries the natural filtrations on both sides onto each other. In particular, if R is quasi-regular semiperfect, $M_{\mathbb{A}}(G)$

is isomorphic to the evaluation $M^{\mathrm{crys}}(G)$ on $A_{\mathrm{crys}}(R)$ of the crystalline Dieudonné crystal, compatibly with the Frobenius and the filtration.

Of course, the isomorphism is linear over the isomorphism $\mathcal{O}^{\mathrm{pris}} \cong \mathcal{O}^{\mathrm{crys}}$ from Lemma 4.3.1.

Proof. By definition

$$\mathcal{M}_{\mathbb{\Delta}}(G) = \mathcal{E}xt^1_{(R)_{\mathrm{qsyn}}}(G, \mathcal{O}^{\mathrm{pris}}).$$

Thus by Lemma 4.3.1 it suffices to see

$$v_*^{\mathrm{crys}}(\mathcal{E}xt^1_{(R/\mathbb{Z}_p)_{\mathrm{crys,pr}}}((u^{\mathrm{crys}})^{-1}(G), \mathcal{O}^{\mathrm{crys}})) \cong \mathcal{E}xt^1_{(R)_{\mathrm{pr}}}(G, \mathcal{O}^{\mathrm{crys}})$$

and similarly with $\mathcal{O}^{\mathrm{crys}}$ replaced by $\mathcal{J}^{\mathrm{crys}}$. This statement follows by a similar reasoning as in Lemma 4.2.4 using that $\mathcal{O}^{\mathrm{crys}}$ is p-complete. Lemma 4.3.1 implies then moreover compatibility with Frobenius and filtration. □

Corollary 4.3.3. *Let R be a quasi-syntomic ring with $pR = 0$ and G a p-divisible group over R. There is a canonical isomorphism*

$$\underline{\mathcal{M}}_{\mathbb{\Delta}}(G) \cong \mathrm{DF}_{\mathrm{Spec}(R)}(G)_{|(R)_{\mathrm{qsyn,pr}}}$$

of filtered Dieudonné crystals over R, where the right hand side is restriction of the filtered Dieudonné functor of [37, §9].

Proof. It is enough to produce this isomorphism over quasi-regular semi-perfect rings. It is given in this case by the last theorem, in view of Lau's definition. □

In general, i.e., when p is not necessarily zero in R, one can still relate the prismatic Dieudonné crystal of a p-divisible group to the crystalline Dieudonné crystal, as follows. Let R be a p-complete ring and let D be a p-complete p-torsion free δ-ring with a surjection $D \to R$ whose kernel has divided powers.[23] As the kernel of $D \to R$ has divided powers, the Frobenius on D induces a morphism $R \to D/p$. With this morphism the prism $(D, (p))$ defines an object of the absolute prismatic site $(R)_{\mathbb{\Delta}}$ of R. Via Lemma 4.2.4 it thus makes sense to evaluate the prismatic Dieudonné module of a p-divisible group over R, more precisely v^* of it, on $(D, (p))$.

Lemma 4.3.4. *For every p-divisible group over R there is a natural Frobenius equivariant, filtered isomorphism*

$$v^*(\mathcal{M}_{\mathbb{\Delta}}(G))(D, (p)) \cong \mathbb{D}(G)(D).$$

Here $\mathbb{D}(G)(D)$ denotes the evaluation of the (contravariant, crystalline) Dieudonné crystal of G on the PD-thickening $D \to R$.

Proof. Assume that H is a finite flat group scheme over R. Then H is syntomic over R and there is a canonical isomorphism

$$H^1((H^{(1)}/D)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}) \cong H^1((H/D)_{\mathrm{crys}}, \mathcal{O}_{\mathrm{crys}})$$

by the crystalline comparison for syntomic morphisms (cf. Remark 3.1.8), where $H^{(1)} := H \times_{\mathrm{Spec}(R)} \mathrm{Spec}(D/p)$, and this holds more generally also for products of

[23] We don't require $p^n R = 0$ for some $n \geq 0$.

H over R[24]. This implies that the spectral sequences (cf. Section 4.4) calculating

$$\mathcal{M}_{\Delta}(H)(D,(p)) := \mathcal{E}xt^1(u^{-1}(H), \mathcal{O}_{\Delta})$$

resp. $\mathbb{D}(H)(D)$ are isomorphic (on the E_1-page, which is sufficient). Hence, we obtain the desired natural isomorphism for finite flat group schemes. The proof of Proposition 4.6.5 below[25] shows that writing

$$G = \varinjlim_n G[p^n]$$

and passing to the limit yields a canonical isomorphism

$$\mathcal{M}_{\Delta}(G)(D,(p)) \cong \mathbb{D}(G)(D)$$

for G a p-divisible group over R. □

Remark 4.3.5. The relation between the prismatic and the crystalline Dieudonné functors will mostly be used over a characteristic p perfect field in the rest of this text, and it could be interesting to find a more direct proof of it in this special case, as explained at the end of this section. But it will also be used for comparison with the Scholze-Weinstein functor in the next paragraph and in Section 5.2.

We turn to perfectoid rings. In this case (see Lemma 4.1.15), it is enough to consider the functor M_{Δ}.

The following statement is a special case of a theorem of Fargues ([22], [47]). Let C be a complete algebraically closed extension of $\mathbb{Q}_p$. We abbreviate

$$A_{\mathrm{inf}} = A_{\mathrm{inf}}(\mathcal{O}_C)\ ,\ A_{\mathrm{crys}} := A_{\mathrm{crys}}(\mathcal{O}_C/p).$$

We also fix a compatible system ε of p-th roots of unity, and let $\tilde{\xi} = [p]_q$, where $q = [\varepsilon] - 1$. We identify the initial prism of $(\mathcal{O}_C)_{\Delta}$ with $(A_{\mathrm{inf}}, (\tilde{\xi}))$.

Proposition 4.3.6. *A prismatic Dieudonné module (M, φ_M) over $\mathcal{O}_C$ (i.e., a minuscule Breuil-Kisin-Fargues module) is uniquely determined up to isomorphism by the triple*

$$(T_M, M_{\mathrm{crys}}, \alpha_M),$$

where T_M is the finite free $\mathbb{Z}_p$-module

$$T_M = M[\frac{1}{\tilde{\xi}}]^{\varphi_M = 1},$$

$$M_{\mathrm{crys}} = M \otimes_{A_{\mathrm{inf}}} A_{\mathrm{crys}}$$

is a φ-module over A_{crys} and $\alpha_M : T_M \otimes_{\mathbb{Z}_p} B_{\mathrm{crys}} \simeq M_{\mathrm{crys}} \otimes_{A_{\mathrm{crys}}} B_{\mathrm{crys}}$ is the φ-equivariant isomorphism coming from the natural map $M[\frac{1}{\tilde{\xi}}]^{\varphi_M=1} \to M[\frac{1}{\tilde{\xi}}]$.

Proposition 4.3.7. *Let R be a perfectoid ring. The functor $G \mapsto M_{\Delta}(G)$ from* $\mathrm{BT}(R)$ *to* $\mathrm{DM}(R)$ *coincides with the (naive)*[26] *dual of the functor M^{SW} of* [47, Appendix to Lecture XVII].

[24]Note that $H^1((H/D)_{\mathrm{crys}}, \mathcal{O}_{\mathrm{crys}}) = H^1(((H/p)/D)_{\mathrm{crys}}, \mathcal{O}_{\mathrm{crys}})$. This follows from the computation of crystalline cohomology by a Čech-Alexander complex and the following fact : if A is a $\mathbb{Z}/p^n$-algebra (for some $n > 0$), P a free $\mathbb{Z}_p$-algebra surjecting onto A, the divided power envelopes of $P/p^m \to A$ and $P/p^m \to A/p$ agree for any $m \geq n$: see [4, Theorem I.2.8.2].

[25]Which the reader can check to be independent of the present lemma.

[26]I.e., $\mathrm{Hom}_{A_{\mathrm{inf}}(R)}(-, A_{\mathrm{inf}}(R))$.

Proof. By v-descent (see [47, Theorem 17.5.2]), one can assume that $R = \mathcal{O}_C$, where C is a perfectoid algebraically closed field. If C has characteristic p, the functor M^{SW} is given by the naive dual of the crystalline Dieudonné functor, so we can simply apply Theorem 4.3.2. Therefore, we can assume that C is a complete algebraically closed extension of $\mathbb{Q}_p$. In this case, assume first that $G = X[p^\infty]$, for some formal abelian scheme X over $\mathcal{O}_C$, with rigid generic fiber X^{rig}. The functor M^{SW} sends G to the prismatic Dieudonné module over $\mathcal{O}_C$ dual to $H^1_{A_{\mathrm{inf}}}(X)$: this follows from the definition of $M^{SW}(G)$ ([47, §12.1]), [47, Proposition 14.8.3] and the previous proposition. In particular, in this case, $M^{SW}(G)$ is isomorphic to the (naive) dual to $M_{\Delta}(G)$, by Corollary 4.5.7 and the comparison theorem [12, Theorem 17.2][27]. Moreover, this identification is functorial for morphisms of p-divisible groups of abelian schemes (and not simply for morphisms of abelian schemes) : indeed, let X, X' be two abelian schemes over $\mathcal{O}_C$, and $G = X[p^\infty]$, $H = X'[p^\infty]$, with a morphism $f\colon G \to H$. We want to see that the diagram

$$\begin{tikzcd}
M^{SW}(G) \arrow[r, "\cong"] \arrow[d, "M^{SW}(f)"'] & M_{\Delta}(G)^* \arrow[d, "M_{\Delta}(f)^*"] \\
M^{SW}(H) \arrow[r, "\cong"] & M_{\Delta}(H)^*
\end{tikzcd}$$

commutes. This can be checked after base change to A_{crys}. Then, using Lemma 4.3.4, the terms on the top line (resp. on the bottom line) are identified with the covariant crystalline Dieudonné module of G (resp. H), and the horizontal isomorphisms induce the identity, by construction.

Let now G be a general p-divisible group over $\mathcal{O}_C$. There exists a formal abelian scheme X over $\mathcal{O}_C$, such that $X[p^\infty] = G \times \check{G}$ (cf. [47, Proposition 14.8.4]). Let $e\colon X[p^\infty] \to X[p^\infty]$ be the idempotent with kernel G. Then

$$M_{\Delta}(G)^* = \ker(M_{\Delta}(e)^*\colon M_{\Delta}(X[p^\infty])^* \to M_{\Delta}(X[p^\infty])^*)$$

and

$$M^{\mathrm{SW}}(G) = \ker(M^{\mathrm{SW}}(e)\colon M^{\mathrm{SW}}(X[p^\infty]) \to M^{\mathrm{SW}}(X[p^\infty])).$$

By the functoriality explained above we can conclude. □

We obtain the following corollary, which we will need in Section 4.9.

Corollary 4.3.8. *Let R be a perfectoid ring. The prismatic Dieudonné functor M_{Δ} induces an antiequivalence between* $\mathrm{BT}(R)$ *and* $\mathrm{DM}(R)$.

In particular, by Lemma 4.1.15, also the filtered prismatic Dieudonné functor $\underline{M_{\Delta}}$ is an antiequivalence in this case.

Proof. This follows immediatley from the last proposition and [47, Theorem 17.5.2]. Note that the argument of loc. cit. shows that one only needs to prove the equivalence when R is the ring of integers of a perfectoid algebraically closed field, where it is due to Berthelot [3, Theorem 3.4.1] and Scholze-Weinstein [48, Theorem 5.2.1] (in this case, one can even assume that the fraction field of R is spherically complete, and the result is then an easy consequence of results of Fargues : see [48, §5.2]). □

[27] Note that we chose $\tilde{\xi}$ as a generator of the ideal of the prism, so the Frobenius twist in the statement of loc. cit. disappears.

Remark 4.3.9. Let R be a perfectoid ring. The functor M_{Δ} is exact (see below Proposition 4.6.7) and has an exact quasi-inverse (we will provide an argument for this later in Section 5.1 in the case of finite locally free group schemes, which applies verbatim for p-divisible groups).

Let us conclude this section by discussing the case of perfect fields. For a perfect field k, Fontaine [23] was the first to give a uniform definition of a functor from p-divisible groups to (prismatic) Dieudonné modules over k. Let us recall it first, as formulated in [6, §4.1]. If A is a commutative ring, the set $\mathrm{CW}(A)$ of *Witt covectors with values in* A is the set of all family $(a_{-i})_{i\in\mathbb{N}}$ of elements of A such that there exist integers $r, s \geq 0$ such that the ideal J_r generated by the a_{-i}, $i \geq r$, satisfies $J_r^s = 0$. One still denotes by CW the sheaf on the big fpqc site[28] of k associated to the presheaf of Witt covectors. This is an abelian sheaf of $W(k)$-modules, endowed with a Frobenius operator which is semi-linear with respect to the Frobenius on $W(k)$. Fontaines defines :

$$M^{\mathrm{cl}}(G) := \mathrm{Hom}_{(k)_{\mathrm{fpqc}}}(G, \mathrm{CW}).$$

As a corollary of Theorem 4.3.2 and results of Berthelot-Breen-Messing, one gets

Proposition 4.3.10. *Let k be a perfect field, and let G be a p-divisible group over R. One has a canonical $W(k)$-linear Frobenius-equivariant isomorphism*

$$M_{\Delta}(G) \cong M^{\mathrm{cl}}(G).$$

Proof. By construction, the isomorphism of Theorem 4.3.2 is linear over the isomorphism $\Delta_k \simeq A_{\mathrm{crys}}(k)$, which is given by the Frobenius σ of $W(k)$, i.e., it can be seen as a Frobenius-equivariant $W(k)$-linear isomorphism :

$$M_{\Delta}(G) \cong (\sigma^{-1})^* M^{\mathrm{crys}}(G).$$

Composing it with σ^{-1}-pullback of the inverse of the $W(k)$-linear Frobenius-equivariant isomorphism of [6, Theorem 4.2.14], we get the desired isomorphism. □

It would be interesting to get a more direct proof of this corollary. In characteristic p, the prismatic Dieudonné crystal of a p-divisible group admits a description which looks similar to Fontaine's definition.

Definition 4.3.11. Let R be a a quasi-syntomic ring with $pR = 0$. We define the sheaf $\mathcal{Q}$ on $(R)_{\Delta}$ as the quotient :

$$0 \to \mathcal{O}_{\Delta} \to \mathcal{O}_{\Delta}[1/p] \to \mathcal{Q} \to 0.$$

The morphism $\mathcal{O}_{\Delta} \to \mathcal{O}_{\Delta}[1/p]$ is injective since any prism in $(R)_{\Delta}$ is p-torsion free.

Proposition 4.3.12. *Let R be a quasi-syntomic ring with $pR = 0$, and let G be a p-divisible group over R. The canonical exact sequence*

$$0 \to \mathcal{O}_{\Delta} \to \mathcal{O}_{\Delta}[1/p] \to \mathcal{Q} \to 0$$

induces an isomorphism :

$$\mathcal{H}om_{(R)_{\mathrm{qsyn}}}(G, v_*\mathcal{Q}) = v_*\mathcal{H}om_{(R)_{\Delta}}(u^{-1}G, \mathcal{O}_{\Delta}) \cong \mathcal{M}_{\Delta}(G).$$

[28] We could as well use any other topology finer than the Zariski topology.

Proof. First assume that G is a finite locally free group scheme. Then the statement is clear, as

$$R\mathcal{H}om_{(R)_{\mathbb{\Delta}}}(u^{-1}(G), \mathcal{O}_{\mathbb{\Delta}}[1/p]) = 0,$$

because $u^{-1}(G)$ is killed by some power of p, whereas on $\mathcal{O}_{\mathbb{\Delta}}[1/p]$ multiplication by p is invertible. The result for p-divisible groups is deduced by a limit argument. $\square$

This naturally leads to the following question.

Question 4.3.13. *When $R = k$ is a perfect field, what is the relation between the sheaf $v_*\mathcal{Q}$ and the sheaf* CW *of Witt covectors ?*

4.4. **Calculating** Ext**-groups in topoi.** In this section we recall the method of calculating Ext-groups in a topos as presented by Berthelot, Breen, Messing (cf. [5, 2.1.5][29]. Let $\mathfrak{X}$ be a topos and let $G, H \in \mathfrak{X}$ be two abelian groups, i.e., two abelian group objects.

The following theorem is attributed to Deligne in [5]. A proof can be found in [45, Appendix to Lecture IV, Theorem 4.10].

Theorem 4.4.1. *Let $G \in \mathfrak{X}$ be an abelian group. Then there exists a natural functorial (in G) resolution*

$$C(G)_\bullet := (\ldots \to \mathbb{Z}[X_2] \to \mathbb{Z}[X_1] \to \mathbb{Z}[X_0]) \simeq G$$

where each $X_i \in \mathfrak{X}$ is a finite disjoint unions of products of copies G.

Proof. See [5, 2.1.5.] or [45, Appendix to Lecture IV, Theorem 4.10] $\square$

Lemma 4.4.2. *Let $X \in \mathfrak{X}$ be any object and let $\mathcal{F} \in \mathrm{Ab}(\mathfrak{X})$ be an abelian group. Then*

$$R\Gamma(X, \mathcal{F}) \cong R\mathrm{Hom}_{\mathrm{Ab}(\mathfrak{X})}(\mathbb{Z}[X], \mathcal{F}),$$

where $\mathbb{Z}[X]$ denotes the free abelian group on X.

Proof. This follows by deriving the isomorphism $\mathcal{F}(X) \cong \mathrm{Hom}_{\mathrm{Ab}(\mathfrak{X})}(\mathbb{Z}[X], \mathcal{F})$. $\square$

These two results show that the Ext-groups

$$\mathrm{Ext}^i_{\mathrm{Ab}(\mathfrak{X})}(G, H)$$

can, *in principle*, be calculated in terms of the cohomology groups

$$H^i(G \times \ldots \times G, H)$$

for various products $G \times \ldots \times G$. Unfortunately, the construction of the resolution in Theorem 4.4.1 is rather involved. However, the first terms, which are sufficient for our applications, can be made explicit[30]. For example, the first terms can be chosen to be

$$C(G)_0 := \mathbb{Z}[G]$$
$$C(G)_1 := \mathbb{Z}[G^2]$$
$$C(G)_2 := \mathbb{Z}[G^3] \oplus \mathbb{Z}[G^2]$$

with explicit differentials (cf. [5, (2.1.5.2.)]). The stupid filtration of the complex $C(G)_\bullet$ yields a spectral sequence

$$E_1^{i,j} = \mathrm{Ext}^j_{\mathrm{Ab}(\mathfrak{X})}(C(G)_i, \mathcal{F}) \Rightarrow \mathrm{Ext}^{i+j}_{\mathrm{Ab}(\mathfrak{X})}(C(G)_\bullet, \mathcal{F}) \cong \mathrm{Ext}^{i+j}_{\mathrm{Ab}(\mathfrak{X})}(G, \mathcal{F})$$

[29] For simplicity we omit the case of the local Ext-sheaves, which is entirely similar.

[30] By this, we mean that one can construct a functorial (in G) resolution having these terms in the beginning.

and the terms

$$\mathrm{Ext}^i_{\mathrm{Ab}(\mathfrak{X})}(C(G)_j, \mathcal{F})$$

can be calculated using the cohomology. For later use let us make the first terms of the first page of this spectral sequence explicit:

$$\begin{tikzcd}
\cdots \arrow[drr, dashed] & \cdots \arrow[drr, dashed] & \cdots & \cdots \\
H^2(G,\mathcal{F}) \arrow[r, "d_1"] \arrow[drr, dashed] & H^2(G\times G,\mathcal{F}) \arrow[r, "d_2"] \arrow[drr, dashed] & H^2(G\times G,\mathcal{F}) \oplus H^2(G\times G\times G,\mathcal{F}) \arrow[r] & \cdots \\
H^1(G,\mathcal{F}) \arrow[r, "d_1"] \arrow[drr, dashed] & H^1(G\times G,\mathcal{F}) \arrow[r, "d_2"] \arrow[drr, dashed] & H^1(G\times G,\mathcal{F}) \oplus H^1(G\times G\times G,\mathcal{F}) \arrow[r] & \cdots \\
H^0(G,\mathcal{F}) \arrow[r, "d_1"] & H^0(G\times G,\mathcal{F}) \arrow[r, "d_2"] & H^0(G\times G,\mathcal{F}) \oplus H^0(G\times G\times G,\mathcal{F}) \arrow[r] & \cdots
\end{tikzcd}$$

For an element $(x_1, \ldots, x_n) \in G^n$ let us denote by $[x_1, \ldots, x_n] \in \mathbb{Z}[G^n]$ the corresponding element in the group ring $\mathbb{Z}[G^n]$. The morphisms d_1 and d_2 are then induced by

$$\mathbb{Z}[G^2] \to \mathbb{Z}[G],\ [x,y] \mapsto -[x] + [x+y] - [y]$$

for d_1 and

$$\mathbb{Z}[G^2] \to \mathbb{Z}[G],\ [x,y] \mapsto [x,y] - [y,x]$$
$$\mathbb{Z}[G^3] \to \mathbb{Z}[G^2],\ [x,y,z] \mapsto -[y,z] + [x+y,z] - [x,y+z] + [x,y]$$

for d_2 (cf. [5, (2.1.5.2.)]).

4.5. **Prismatic Dieudonné crystals of abelian schemes.** In this section we describe the prismatic cohomology of the p-adic completion of abelian schemes and deduce from this the construction of the filtered prismatic Dieudonné crystal

$$\underline{\mathcal{M}}_{\Delta}(X[p^\infty]) = (\mathcal{M}_{\Delta}(X[p^\infty]), \mathrm{Fil}\mathcal{M}_{\Delta}(X[p^\infty]), \varphi_{\mathcal{M}_{\Delta}(X[p^\infty])}).$$

of the p-divisible group $X[p^\infty]$ of the p-adic completion of an abelian scheme X. This will be done more generally for any p-divisible group over R in the next section, using the results of this section. Nevertheless, for clarity - which yields some redundancy - we decided to discuss the case of p-divisible groups attached to abelian schemes completely first.

Let (A, I) be a bounded prism and let $X \to \mathrm{Spf}(A/I)$ be the p-adic completion of an abelian scheme over $\mathrm{Spec}(A/I)$.

We first prove degeneracy of the conjugate spectral sequence (cf. Proposition 3.1.10) for X. The proof is an adaption of the argument in [5, Proposition 2.5.2.], which proves degeneration of the Hodge-de Rham spectral sequence.

Recall the following statement.

Proposition 4.5.1. *For all $k \geq 0$ (resp. for all $i, j \geq 0$), the A/I-module $H^k(X, \Omega^\bullet_{X/(A/I)})$ (resp. $H^i(X, \Omega^j_{X/(A/I)})$) is finite locally free, and its formation commutes with base change.*

Moreover, the algebra $H^*(X, \Omega^\bullet_{X/(A/I)})$ is alternating and the canonical algebra morphism

$$\wedge^* H^1(X, \Omega^\bullet_{X/(A/I)}) \to H^*(X, \Omega^\bullet_{X/(A/I)})$$

defined by the multiplicative structure of $H^*(X, \Omega^\bullet_{X/(A/I)})$, *is an isomorphism.*

Proof. This is [5, Proposition 2.5.2. (i)-(ii)]. □

Proposition 4.5.2. *The conjugate spectral sequence*

$$E_2^{ij} = H^i(X, \Omega^j_{X/A/I})\{-j\} \Rightarrow H^{i+j}(X, \overline{\mathbb{\Delta}}_{X/A})$$

degenerates and each term as well as the abutment commutes with base change in the bounded prism (A, I). *Moreover,*

$$H^*(X, \overline{\mathbb{\Delta}}_{X/A}) \cong \Lambda^* H^1(X, \overline{\mathbb{\Delta}}_{X/A})$$

is an exterior A/I*-algebra on* $H^1(X, \overline{\mathbb{\Delta}}_{X/A})$.

Proof. Let n be the relative dimension of X over $\mathrm{Spf}(A/I)$. By Proposition 4.5.1, for every $i, j \geq 0$ the cohomology group

$$H^i(X, \Omega^j_{X/A/I})\{-j\}$$

is a locally free A/I-modules of finite rank and commutes with base change in A/I. Moreover, $H^i(X, \Omega^j_{X/A/I})\{-j\} = 0$ if $i + j > n$. We argue by descending induction that the A/I-modules

$$H^k(X, \overline{\mathbb{\Delta}}_{X/A})$$

are locally free of finite rank. If $k = n$, then

$$H^n(X, \overline{\mathbb{\Delta}}_{X/A}) \cong H^n(X, \Omega^n_{X/A/I})\{-n\} \xrightarrow[\simeq]{\mathrm{tr}\{-n\}} A/I\{-n\}$$

because X is the p-adic completion of a proper, smooth scheme of relative dimension n with geometrically connected fibers. Now assume that the claim is proven for all $i > k$. We will show that it suffices to prove that $H^1(X, \overline{\mathbb{\Delta}}_{X/A})$ is locally free of finite rank over A/I and commutes with base change in (A, I). We assume for the moment that this is true. There is a canonical morphism

$$\wedge^i H^1(X, \overline{\mathbb{\Delta}}_{X/A}) \to H^i(X, \overline{\mathbb{\Delta}}_{(X/A)})$$

for all i and a pairing

$$H^{n-k}(X, \overline{\mathbb{\Delta}}_{X/A}) \otimes_{A/I} H^k(X, \overline{\mathbb{\Delta}}_{X/A}) \to H^n(X, \overline{\mathbb{\Delta}}_{X/A}) \cong A/I$$

induced by multiplication in $H^*(X, \overline{\mathbb{\Delta}}_{X/A})$ These two morphisms yield a canonical morphism

$$\beta\colon H^k(X, \overline{\mathbb{\Delta}}_{X/A}) \to (\wedge^{n-k}(H^1(X, \overline{\mathbb{\Delta}}_{X/A})))^\vee \cong \wedge^k(H^1(X, \overline{\mathbb{\Delta}}_{X/A})).$$

By the Hodge-Tate comparison the complex

$$\overline{\mathbb{\Delta}}_{X/A}$$

satisfies base change in (A, I), i.e., for a morphism $(A, I) \to (A', I')$ of prisms with induced morphism $g\colon X' := X \times_{\mathrm{Spf}(A/I)} \mathrm{Spf}(A'/I') \to X$ the canonical morphism

$$Lg^* \overline{\mathbb{\Delta}}_{X/A} \to \overline{\mathbb{\Delta}}_{X'/A'}$$

is an isomorphism. By induction the cohomology groups

$$H^i(X, \overline{\mathbb{\Delta}}_{X/A})$$

are locally free of finite rank for $i > k$. This implies that the cohomology group

$$H^k(X, \overline{\mathbb{\Delta}}_{X/A})$$

commutes with base change in the prism (A', I') as well. We want to show that the morphism

$$\beta\colon H^k(X, \overline{\Delta}_{X/A}) \to (\wedge^{n-k}(H^1(X, \overline{\Delta}_{X/A})))^\vee \cong \wedge^k(H^1(X, \overline{\Delta}_{X/A})).$$

is an isomorphism. The cohomology group $H^k(X, \overline{\Delta}_{X/A})$ is finitely presented over A/I because $X \to \mathrm{Spf}(A/I)$ is proper and flat, $\overline{\Delta}_{X/A}$ is a perfect complex on X and all $H^i(X, \overline{\Delta}_{X/A})$ for $i > k$ are locally free of finite rank. Thus we may apply Lemma 4.5.3 and, after base change to the algebraic closures of the residue fields of A/p, assume that A/I is an algebraically closed field of characteristic p. In particular, the Frobenius on A is bijective in this case, $I = (p)$ and the twists $(-)\{j\}$ are isomorphic to the identity. It suffices to show that for all k the cohomology group $H^k(X, \overline{\Delta}_{X/A})$ has the correct dimension. This may be checked after base change along $\varphi_{A/I}$. Then

$$\varphi_{A/I}^* H^k(X, \overline{\Delta}_{X/A}) \cong H^k(X^{(1)}, (\varphi_{X/A/I})_*(\Omega^\bullet_{X/A/I})) \cong H^k(X, \Omega^\bullet_{X/A/I})$$

where we used in the second isomorphism that the relative Frobenius

$$\varphi_{X/A/I}\colon X \to X^{(1)} := X \times_{\mathrm{Spec}(A/I), \varphi_{A/I}} \mathrm{Spec}(A/I)$$

is finite. By Proposition 4.5.1 this shows that this cohomology group has the correct dimension. Thus we have reduced the proof to showing that $H^1(X, \overline{\Delta}_{X/A})$ is locally free of finite rank and commutes with base change in (A, I). From Proposition 3.2.1 it follows that

$$H^1(X, \overline{\Delta}_{X/A}) \cong H^1(X, \tau_{\leq 1}\overline{\Delta}_{X/A}) \cong H^0(X, L_{X/A}[-1]).$$

as $L_{X/A}$ is a perfect complex with amplitude in $[-1, 0]$ this implies compatibility of $H^1(X, \overline{\Delta}_{X/A})$ with base change in (A, I) if all the higher cohomology grous $H^j(X, L_{X/A}[-1])$ are locally free. As X admits a lift to A (see e.g. [44, Theorem 2.2.1]), Proposition 3.2.2 shows that $L_{X/A} \cong \mathcal{O}_X[1] \oplus \Omega^1_{X/A/I}$. Another application of Proposition 4.5.1 implies therefore that $H^1(X, \overline{\Delta}_{X/A})$ is locally free of dimension $2n$ (and commutes with base change in (A, I) as all the A/I-modules $H^j(X, \mathcal{O}_X)$ and $H^j(X, \Omega^1_{X/A/I})$ are locally free for $j \geq 0$. □

Lemma 4.5.3. *Let S be a ring and let $g\colon M \to N$ be a morphism of S-modules with M finitely generated and N finite projective. If*

$$g \otimes_S k(x)\colon M \otimes_S k(x) \to N \otimes_S k(x)$$

is an isomorphism for all closed points $x \in \mathrm{Spec}(S)$, then g is an isomorphism.

Proof. Let Q be the cokernel of g. Then Q is finitely generated and $Q \otimes_S k(x) = 0$ for all closed points $x \in \mathrm{Spec}(S)$. By Nakayama's lemma, this implies that $Q = 0$, i.e., g is surjective. As N is projective, this implies $M \cong N \oplus K$ for K the kernel of g. As M is finitely generated, K is finitely generated. Moreover for all closed points $x \in \mathrm{Spec}(S)$

$$K \otimes_S k(x) = 0$$

and thus another application of Nakayama's lemma implies that $K = 0$. □

We recall that for a p-complete ring R there is the natural morphism of topoi

$$u\colon \mathrm{Shv}(R)_{\Delta} \to \mathrm{Shv}(R)_{\mathrm{QSYN}}.$$

Using the previous computations, we can first describe extension groups modulo I.

Theorem 4.5.4. *Let R be a p-complete ring and let $f\colon X \to \mathrm{Spf}(R)$ be the p-adic completion of an abelian scheme over $\mathrm{Spec}(R)$. Then*

(1) $\mathcal{E}xt^i_{(R)_{\Delta}}(u^{-1}(X), \overline{\mathcal{O}}_{\Delta}) = 0$ *for $i = 0, 2$.*

(2) $\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}(X), \overline{\mathcal{O}}_{\Delta})$ *is a prismatic crystal over R. Moreover,*

$$\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}(X), \overline{\mathcal{O}}_{\Delta}) \cong R^1 f_{\Delta,*}(\overline{\mathcal{O}}_{\Delta})$$

for $f_{\Delta}\colon \mathrm{Shv}(X)_{\Delta} \to \mathrm{Shv}(R)_{\Delta}$ the morphism induced by f on topoi and $\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}(X), \overline{\mathcal{O}}_{\Delta})$ is locally free of rank $2\dim(X)$ over $\overline{\mathcal{O}}_{\Delta}$.

The proof is entirely similar to the one of [5, Théorème 2.5.6.]

Proof. Let $(B, J) \in (R)_{\Delta}$. We use the spectral sequence from Section 4.4 to calculate for $i \in \{0, 1, 2\}$ the groups

$$\mathrm{Ext}^i(u^{-1}(X)_{|(B,J)}, \overline{\mathcal{O}}_{\Delta})$$

on the localised site $(R)_{\Delta}/(B, J)$[31]. Set $Y := X \times_{\mathrm{Spf}(A/I)} \mathrm{Spf}(B/J)$. As by Hodge-Tate comparison

$$H^0(Y, \overline{\Delta}_{Y/B}) \cong B/J$$

for any n the first row of the spectral sequence is isomorphic to

$$B/J \xrightarrow{\mathrm{Id}_{B/J}} B/J \xrightarrow{0} B/J^2 \xrightarrow{\alpha} B/J^2 \oplus B/J^3$$

with $\alpha(x, y) = (x, x + y, -x + y, -2y, -y)$. Thus $\mathrm{Hom}(u^{-1}(X)_{|(B,J)}, \overline{\mathcal{O}}_{\Delta}) = 0$ and $\mathrm{Ext}^1(u^{-1}(X)_{|(B,J)}, \overline{\mathcal{O}}_{\Delta})$ is isomorphic to the kernel of

$$H^1(Y, \overline{\Delta}_{Y/B}) \xrightarrow{d_1} H^1(Y, \overline{\Delta}_{Y/B})$$

and $d_1 = \mathrm{pr}_1^* + \mathrm{pr}_2^* - \mu^*$ for pr_i the two projections and μ the multiplication. From the Künneth formula (cf. Corollary 3.5.2) and Corollary 4.5.8 it follows that

$$H^1(Y, \overline{\Delta}_{Y/B}) \cong H^1(Y, \overline{\Delta}_{Y/B}) \oplus H^1(Y, \overline{\Delta}_{Y/B}).$$

This implies $\mu^* = \mathrm{pr}_1^* + \mathrm{pr}_2^*$, i.e., $d_1 = 0$ and

$$\mathrm{Ext}^1(u^{-1}(X)_{|(B,J)}, \overline{\mathcal{O}}_{\Delta}) \cong H^1(Y, \overline{\Delta}_{Y/B}).$$

In particular, this group is compatible with base change in (B, J) and locally free of rank $2\dim(X)$ (by Proposition 4.5.2). Moreover, the morphism d_2 is injective on $H^1(Y, \overline{\Delta}_{Y/B})$ as follows from the Künneth theorem and the concrete formula for d_2. Finally, from Corollary 4.5.8 and Lemma 4.5.5 one can deduce that

$$H^i(Y, \overline{\Delta}_{Y/B}) \xrightarrow{d_1} H^i(Y, \overline{\Delta}_{Y/B})$$

is injective for all $i \geq 2$. These statements implies $\mathrm{Ext}^2(u^{-1}(X)_{|(B,J)}, \overline{\mathcal{O}}_{\Delta}) = 0$. This finishes the proof by passing to the local Ext-groups, i.e., by letting (B, J) vary. □

In the proof we used the following lemma on primitive elements in exterior algebras.

[31] Which will be implicitly the subscript of all Ext-groups appearing in this proof.

Lemma 4.5.5. *Let S be a ring and let M be a projective S-module. Then*

$$\{x \in \Lambda(M) \mid \mu^*(x) = 1 \otimes x + x \otimes 1\} = \Lambda^1 M,$$

where $\mu^* \colon \Lambda(M) \to \Lambda(M+M) \cong \Lambda(M) \otimes_S \Lambda(M)$ is the natural comultiplication on $\Lambda(M)$ coming from the diagonal $M \to M \oplus M$.

Proof. This follows easily by decomposing $\Lambda(M) \otimes_S \Lambda(M)$ into its bigraded pieces $\Lambda^i(M) \otimes_S \Lambda^j(M)$. ☐

Now we calculate the full extension groups, up to degree 2.

Theorem 4.5.6. *Let R be a p-complete ring and let $f \colon X \to \mathrm{Spf}(R)$ be the p-adic completion of an abelian scheme over* $\mathrm{Spec}(R)$. *Then*

(1) $\mathcal{E}xt^i_{(R)_{\Delta}}(u^{-1}(X), \mathcal{O}_{\Delta}) = 0$ *for $i = 0, 2$.*

(2) $\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}(X), \mathcal{O}_{\Delta})$ *is a prismatic crystal over R. Moreover,*

$$\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}(X), \mathcal{O}_{\Delta}) \cong R^1 f_{\Delta,*}(\mathcal{O}_{\Delta}),$$

for $f_{\Delta} \colon \mathrm{Shv}(X)_{\Delta} \to \mathrm{Shv}(R)_{\Delta}$ the induced morphism on topoi and the prismatic crystal $\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}(X), \mathcal{O}_{\Delta})$ is locally free of rank $2\mathrm{dim}(X)$ *over A/I.*

Proof. Let $(B, J) \in (R)_{\Delta}$. As the statements are local for the faithfully flat topology we may assume that $J = (\tilde{\xi})$ is principal. From the exact sequence

$$0 \to \mathcal{O}_{\Delta}/\tilde{\xi}^n \xrightarrow{\tilde{\xi}} \mathcal{O}_{\Delta}/\tilde{\xi}^{n+1} \to \mathcal{O}_{\Delta}/\tilde{\xi} = \overline{\mathcal{O}}_{\Delta} \to 0$$

of sheaves on $(R)_{\Delta}/(B, J)$ and Theorem 4.5.4 we can inductively conclude that

$$\mathrm{Ext}^i(u^{-1}(X)_{|(B,J)}, \mathcal{O}_{\Delta}/(\tilde{\xi}^n)) = 0$$

for $i \in \{0, 2\}$ and any $n \geq 0$. This implies that

$$0 \to \mathrm{Ext}^1(u^{-1}(X)_{|(B,J)}, \mathcal{O}_{\Delta}/(\tilde{\xi}^n)) \xrightarrow{\tilde{\xi}} \mathrm{Ext}^1(u^{-1}(X)_{|(B,J)}, \mathcal{O}_{\Delta}/(\tilde{\xi}^{n+1})) \\ \to \mathrm{Ext}^1(u^{-1}(X)_{|(B,J)}, \overline{\mathcal{O}}_{\Delta}) \to 0$$

is exact and that

$$\mathrm{Ext}^i(u^{-1}(X)_{|(B,J)}, \mathcal{O}_{\Delta}) \cong \varprojlim_n \mathrm{Ext}^i(u^{-1}(X)_{|(B,J)}, \mathcal{O}_{\Delta}/(\tilde{\xi}^n)),$$

and that it is zero for $i \in \{0, 2\}$ or a locally free B-module of rank $2\mathrm{dim}(X)$ if $i = 1$. Using the spectral sequence from Section 4.4 we can see similarly to Theorem 4.5.4 that

$$\mathrm{Ext}^1(u^{-1}(X)_{|(B,J)}, \mathcal{O}_{\Delta}) \cong H^1(X \times_{\mathrm{Spf}(R)} \mathrm{Spf}(B/J), \Delta_{X/A}).$$

This finishes the proof by passing to local Ext-groups. ☐

Corollary 4.5.7. *Let R be a p-complete ring. Let X be the p-completion of an abelian scheme over R. The $\mathcal{O}^{\mathrm{pris}}$-module*

$$\mathcal{M}_{\Delta}(X[p^\infty]) = \mathcal{E}xt^1_{(R)_{\mathrm{qsyn}}}(G, \mathcal{O}^{\mathrm{pris}})$$

is a finite locally free $\mathcal{O}^{\mathrm{pris}}$-module of rank $2\dim(X)$, given by $R^1 f_{\Delta,*}\mathcal{O}_{\Delta}$.

Proof. By Lemma 4.2.4,

$$\mathcal{M}_{\Delta}(X[p^\infty]) = v_*(\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G, \mathcal{O}_{\Delta})).$$

Hence the corollary results from Theorem 4.5.6 and Proposition 4.1.4. ☐

Although we will not use it, let us record the full description of the prismatic cohomology of X.

Corollary 4.5.8. *With the notation from Corollary 4.5.7, the prismatic cohomology*

$$R^* f_{\mathbb{\Delta},*}\mathcal{O}_{\mathbb{\Delta}}$$

is a finite locally free crystal on $(R)_{\mathbb{\Delta}}$ and an exterior algebra on the locally free crystal

$$R^1 f_{\mathbb{\Delta},*}(\mathcal{O}_{\mathbb{\Delta}})$$

of dimension $2\dim(X)$.

Proof. Let $(B,J)\in(R)_{\mathbb{\Delta}}$ and let $Y := X\times_{\mathrm{Spf}(R)}\mathrm{Spf}(B/J)$. It suffices to prove the analog statements for $H^*(Y,\mathbb{\Delta}_{Y/B})$. From (the proof of) Theorem 4.5.6 we see that

$$H^1(Y,\mathbb{\Delta}_{Y/B})\to H^1(Y,\overline{\mathbb{\Delta}}_{Y/B})$$

is surjective and that $H^*(Y,\overline{\mathbb{\Delta}}_{Y/B})$ is an exterior algebra on $H^1(Y,\overline{\mathbb{\Delta}}_{Y/B})$. Let $g := \dim Y$. By lifting $2g$-generators of $H^1(Y,\overline{\mathbb{\Delta}}_{Y/B})$ we obtain a morphism

$$B^{2g}[-1]\to\mathbb{\Delta}_{Y/B}.$$

Using the alternating products in $H^*(Y,\mathbb{\Delta}_{Y/B})$ of the images of the standard basis of B^{2g} we obtain a morphism

$$\Lambda^i B^{2g}[-i]\to R\Gamma(Y,\mathbb{\Delta}_{Y/B})$$

inducing an isomorphism on H^i after passing to $\otimes^{\mathbb{L}}_B B/J$. Altogether, we obtain a morphism

$$\Lambda^*(B^{2g})[-*]\to R\Gamma(Y,\mathbb{\Delta}_{Y/B})$$

of complexes which is an isomorphism after applying $\otimes^{\mathbb{L}}_B B/J$. By derived J-adic completeness it is therefore an isomorphism, which implies the statements. □

We now turn to the filtration on the prismatic Dieudonné crystal of our abelian scheme. From now on, the base ring R will be assumed to be quasi-syntomic.

Proposition 4.5.9. *Let R be a quasi-syntomic ring and let $f\colon X\to\mathrm{Spf}(R)$ be the p-completion of an abelian scheme. Then there is a canonical exact sequence :*

$$0\to\mathrm{Fil}\mathcal{M}_{\mathbb{\Delta}}(X[p^\infty])\to\mathcal{M}_{\mathbb{\Delta}}(X[p^\infty])\to R^1 f_{\mathrm{qsyn},*}\mathcal{O}\to 0,$$

where $f_{\mathrm{qsyn}}\colon X_{\mathrm{qsyn}}\to\mathrm{Spf}(R)_{\mathrm{qsyn}}$ is the natural morphism of sites.

Proof. Applying $R\mathcal{H}om_{(R)_{\mathrm{qsyn}}}(X,-)$ to the short exact sequence

$$0\to\mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}\to\mathcal{O}^{\mathrm{pris}}\to\mathcal{O}\to 0$$

on $(X)_{\mathrm{qsyn}}$ (cf. Proposition 4.1.2), one gets a sequence

$$0\to\mathrm{Fil}\mathcal{M}_{\mathbb{\Delta}}(X[p^\infty])\to\mathcal{M}_{\mathbb{\Delta}}(X[p^\infty])\to\mathcal{E}xt^1_{(R)_{\mathrm{qsyn}}}(X,\mathcal{O})\cong R^1 f_{\mathrm{qsyn},*}\mathcal{O}\to 0$$

and we need to prove that this sequence is left and right exact[32]. Left exactness follows from the fact that

$$\mathcal{H}om_{(R)_{\mathrm{qsyn}}}(X,\mathcal{O})=0.$$

[32]The isomorphism $\mathcal{E}xt^1_{(R)_{\mathrm{qsyn}}}(X,\mathcal{O})\cong R^1 f_{\mathrm{qsyn},*}\mathcal{O}$ follows from a similar reasoning as in Theorem 4.5.4 (because one can use that $R^* f_{\mathrm{qsyn},*}\mathcal{O}$ is an exterior algebra on $R^1 f_{\mathrm{qsyn},*}\mathcal{O}$).

To prove right exactness, we can assume that R is quasi-regular semiperfectoid, by Proposition 3.3.7. We claim that in this case, the evaluation on R of the above map

$$M_{\mathbb{\Delta}}(X[p^\infty]) \to H^1(X, \mathcal{O})$$

is surjective. The target of this $\mathbb{\Delta}_R$-linear map is a finitely generated $\mathbb{\Delta}_R$-module (even, finiteley generated R-module), so by Nakayama's lemma and p-completeness of R, we can check surjectivity after base change along each surjective map $\mathbb{\Delta}_R \to k$, where k is a characteristic p perfect field. Any such map extends to a map of δ-rings $\mathbb{\Delta}_R \to W(k)$. Moreover, the base change of our map

$$M_{\mathbb{\Delta}}(X[p^\infty]) \to H^1(X, \mathcal{O})$$

along $\mathbb{\Delta}_R \to W(k)$ is the corresponding map for the abelian variety $X \times_R k$ over k. Thus we can assume that $R = k$ is a characteristic p perfect field. In this case, the result is a consequence of the comparison with the crystalline functor (cf. Theorem 4.3.2) and [5, Proposition 2.5.8]. □

Proposition 4.5.10. *Let R be a quasi-syntomic ring, and let X be the p-completion of an abelian scheme over R. The triple $\underline{\mathcal{M}}_{\mathbb{\Delta}}(X[p^\infty])$ of Definition 4.2.8 is a filtered Dieudonné crystal over R.*

Proof. We check the conditions of Definition 4.1.6. By Corollary 4.5.7, $\mathcal{M}_{\mathbb{\Delta}}(X[p^\infty])$ is a finite locally free $\mathcal{O}^{\mathrm{pris}}$-module. Proposition 4.5.9 shows that $\mathrm{Fil}\mathcal{M}_{\mathbb{\Delta}}(X[p^\infty])$ is indeed a submodule of $\mathcal{M}_{\mathbb{\Delta}}(X[p^\infty])$, which contains $\mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}.\mathcal{M}_{\mathbb{\Delta}}(X[p^\infty])$, and by Lemma 4.2.3,

$$\varphi_{\mathcal{M}_{\mathbb{\Delta}}(X[p^\infty])}(\mathrm{Fil}\mathcal{M}_{\mathbb{\Delta}}(X[p^\infty])) \subset \mathcal{I}^{\mathrm{pris}}\mathcal{M}_{\mathbb{\Delta}}(X[p^\infty]).$$

It thus only remains to verify the second part of Condition (2) and Condition (3) of Definition 4.1.10.

Proposition 4.5.9 shows that the quotient

$$\mathcal{M}_{\mathbb{\Delta}}(X[p^\infty])/\mathrm{Fil}\mathcal{M}_{\mathbb{\Delta}}(X[p^\infty]) \simeq R^1 f_{\mathrm{qsyn},*}\mathcal{O},$$

which is a finite locally free $\mathcal{O}$-module (cf. Proposition 4.5.1). Hence the second requirement of Condition (2) is fullfilled. To check Condition (3), we can assume that R is quasi-regular semiperfectoid. This condition says that the linearization of the Frobenius

$$\varphi^*\mathrm{Fil}M_{\mathbb{\Delta}}(X[p^\infty]) \to I.M_{\mathbb{\Delta}}(X[p^\infty])$$

is an isomorphism. Exactly as in the proof of Proposition 4.5.9, this can be checked when $R = k$ is a characteristic p perfect field, in which case this is known by comparison with the crystalline functor (cf. Theorem 4.3.2). □

4.6. **The filtered prismatic Dieudonné crystal of a p-divisible group.** In this section, we establish the basic properties of the filtered prismatic Dieudonné functor for p-divisible groups. The idea, due to Berthelot-Breen-Messing, is to make systematic use of the following theorem of Raynaud, to reduce to statements about (p-divisible groups of) abelian schemes proved in the last section.

Theorem 4.6.1. *Let S be a scheme, and let G be a finite locally free group scheme over S. There exists Zariski-locally on S, a (projective) abelian scheme A and a closed immersion $G \hookrightarrow A$ over S.*

Proof. See [5, Theorem 3.1.1]. □

Proposition 4.6.2. *Let R be a p-complete ring, and let G be a finite locally free group scheme over R. The sheaf $\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G, \mathcal{O}_{\Delta})$ is a prismatic crystal of locally finitely presented $\mathcal{O}_{\Delta}$-modules.*

Proof. By Theorem 4.6.1, one can choose locally on R an exact sequence of group schemes

$$0 \to G \to X \to X' \to 0,$$

where X and X' are abelian schemes over R. Whence, by Theorem 4.5.6 (1), an exact sequence

$$\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}X', \mathcal{O}_{\Delta}) \to \mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}X, \mathcal{O}_{\Delta}) \to \mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G, \mathcal{O}_{\Delta}) \to 0.$$

This proves the proposition, by Theorem 4.5.6 (2). □

We recall that a finite locally free group scheme G over a scheme S is called Barsotti-Tate of level $n \geq 0$ if, Zariski-locally on S, there exists an isomorphism $G \cong H[p^n]$ for a p-divisible group H over S.

Remark 4.6.3. Let G be a finite locally free group scheme over a basis on which p is nilpotent, and let ℓ_G be its coLie complex. Set :

$$\omega_G = H^0(\ell_G)\ ,\ n_G = H^{-1}(\ell_G)\ ,\ t_G = H^0(\check{\ell}_G)\ ;\ \nu_G = H^1(\check{\ell}_G).$$

Grothendieck's duality formula identifies $\check{\ell}_G$ with the truncation $\tau^{\leq 1} R\mathcal{H}om(G^*, \mathbb{G}_a)$, and this gives rise to a canonical morphism :

$$\phi_G : \nu_G \to t_G.$$

If G is killed by p^n, then G is a BT_n if and only if t_G, t_{G^*} are locally free and the canonical morphisms ϕ_G and ϕ_{G^*} are isomorphisms (cf. [25, Corollary 2.2.5]).

Proposition 4.6.4. *Let R be a p-complete ring, and let G be a truncated Barsotti-Tate group over R of level n. The sheaf $\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G, \mathcal{O}_{\Delta})$ is a prismatic crystal of finite locally free $\mathcal{O}_{\Delta}/p^n$-modules.*

Proof. Fix once and for all an embedding of G into an abelian scheme X of dimension g over R. By Theorem 4.6.1, this can be done Zariski-locally on R, and the reader can check that the different steps of the proof are all local statements on R. Let X' be the cokernel of the embedding $G \to X$; this an abelian scheme, and one has an exact sequence

$$0 \to G \to X \to X' \to 0$$

of group schemes over R.

We first prove that for any $(B, J) \in (R)_{\Delta}$, the B-module

$$\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G, \mathcal{O}_{\Delta})_{(B,J)}$$

is locally generated by h sections, where h is the height of G. By the crystal property of $\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G, \mathcal{O}_{\Delta})$ (cf. Proposition 4.6.2), for any morphism of prisms $(B, J) \to (W(k), (p))$, where k is a characteristic p perfect field,

$$\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G, \mathcal{O}_{\Delta})_{(B,J)} \otimes_B W(k) = \mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G_k, \mathcal{O}_{\Delta})_{(W(k),(p))}.$$

By Nakayama's lemma, p-completeness of B and the finite presentation proved in Proposition 4.6.2, it suffices to prove that for any morphism $B \to k$, k characteristic p perfect field,

$$\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G, \mathcal{O}_{\Delta})_{(B,J)} \otimes_B k$$

is generated by h elements. Such a morphism $B \to k$ extends to a morphism of prisms $(B, J) \to (W(k), (p))$, so it suffices by the above to prove our claim when $R = k$ is a perfect field and $(B, J) = (W(k), (p))$. First, observe that

$$\mathcal{E}xt^1_{(k)_{\mathbb{\Delta}}}(u^{-1}G, \mathcal{O}_{\mathbb{\Delta}})_{(W(k),(p))} \otimes k = \mathcal{E}xt^1_{(k)_{\mathbb{\Delta}}}(u^{-1}G, \overline{\mathcal{O}}_{\mathbb{\Delta}})_{(W(k),(p))}.$$

This is easily seen, using that $\mathcal{E}xt^2_{(k)_{\mathbb{\Delta}}}(u^{-1}X, \mathcal{O}_{\mathbb{\Delta}})$ and $\mathcal{E}xt^2_{(k)_{\mathbb{\Delta}}}(u^{-1}X, \overline{\mathcal{O}}_{\mathbb{\Delta}})$ both vanish.

As a corollary of Proposition 4.5.2 and Theorem 4.5.4, one has a short exact sequence

$$0 \to u^*\mathrm{Lie}(X^*) \to \mathcal{E}xt^1_{(R)_{\mathbb{\Delta}}}(u^{-1}X, \overline{\mathcal{O}}_{\mathbb{\Delta}}) \to u^*\omega_X \to 0,$$

and similarly for X'. Also, note that we have exact sequences[33] :

$$u^*\mathrm{Lie}(X^*) \to u^*\mathrm{Lie}(X^{'*}) \to u^*\nu_{G^*} \to 0$$

(where $\nu_{G^*} = \mathcal{E}xt^1(G, \mathbb{G}_a)$) and

$$u^*\omega_X \to u^*\omega_{X'} \to u^*\omega_G \to 0.$$

The map $\mathcal{E}xt^1_{(k)_{\mathbb{\Delta}}}(u^{-1}X', \overline{\mathcal{O}}_{\mathbb{\Delta}}) \to \mathcal{E}xt^1_{(k)_{\mathbb{\Delta}}}(u^{-1}X, \overline{\mathcal{O}}_{\mathbb{\Delta}})$ is compatible with the natural maps $u^*\mathrm{Lie}(X^*) \to u^*\mathrm{Lie}(X^{'*})$ and $u^*\omega_{X'} \to u^*\omega_X$, through the identifications of Theorem 4.5.4. The long exact sequence of $\mathcal{E}xt$ gives a surjection :

$$\mathcal{E}xt^1_{(k)_{\mathbb{\Delta}}}(u^{-1}X', \overline{\mathcal{O}}_{\mathbb{\Delta}}) \to \mathcal{E}xt^1_{(k)_{\mathbb{\Delta}}}(u^{-1}X, \overline{\mathcal{O}}_{\mathbb{\Delta}}) \to \mathcal{E}xt^1_{(k)_{\mathbb{\Delta}}}(u^{-1}G, \overline{\mathcal{O}}_{\mathbb{\Delta}}) \to 0,$$

since, as we have seen in Theorem 4.5.4, $\mathcal{E}xt^2_{(k)_{\mathbb{\Delta}}}(u^{-1}X', \overline{\mathcal{O}}_{\mathbb{\Delta}}) = 0$. By the above remark, we even have a commutative diagram :

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] & \\
u^*\mathrm{Lie}(X^*) \arrow[r] \arrow[d] & u^*\mathrm{Lie}(X^{'*}) \arrow[r] \arrow[d] & u^*\nu_{G^*} \arrow[r] \arrow[d] & 0 \\
\mathcal{E}xt^1_{(k)_{\mathbb{\Delta}}}(u^{-1}X, \overline{\mathcal{O}}_{\mathbb{\Delta}}) \arrow[r] \arrow[d] & \mathcal{E}xt^1_{(k)_{\mathbb{\Delta}}}(u^{-1}X', \overline{\mathcal{O}}_{\mathbb{\Delta}}) \arrow[r] \arrow[d] & \mathcal{E}xt^1_{(k)_{\mathbb{\Delta}}}(u^{-1}G, \overline{\mathcal{O}}_{\mathbb{\Delta}}) \arrow[r] \arrow[d] & 0 \\
u^*\omega_X \arrow[r] \arrow[d] & u^*\omega_{X'} \arrow[r] \arrow[d] & u^*\omega_G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & 0 &
\end{tikzcd}$$

where all lines and the first two columns are exact. This proves that the map

$$\mathcal{E}xt^1_{(k)_{\mathbb{\Delta}}}(u^{-1}G, \overline{\mathcal{O}}_{\mathbb{\Delta}}) \to u^*\omega_G$$

is surjective and an easy diagram chase prove that in fact the sequence

$$u^*\nu_{G^*} \to \mathcal{E}xt^1_{(k)_{\mathbb{\Delta}}}(u^{-1}G, \overline{\mathcal{O}}_{\mathbb{\Delta}}) \to u^*\omega_G \to 0$$

is exact. The sheaf ω_G is a locally free sheaf of rank $d = \dim G$. Moreover, as G is a truncated Barsotti-Tate group, ν_{G^*} is a locally free sheaf of rank $h - d$ (cf. [25,

[33] Recall ([5, §5.1.1]) that if X is an abelian scheme, $\mathrm{Lie}(X^*) \cong \mathrm{Ext}^1(X, \mathbb{G}_a)$.

Corollary 2.2.5]). Hence the sequence stays exact after evaluation on $(W(k),(p))$ and $\mathcal{E}xt^1_{(k)_{\Delta}}(u^{-1}G,\overline{\mathcal{O}}_{\Delta})_{(W(k),(p))}$ is generated by h sections. This proves the claim.

Back to the proof of the proposition, we know, as a direct consequence of Theorem 4.5.6 that

$$\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}X[p^n],\mathcal{O}_{\Delta}) = \mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}X,\mathcal{O}_{\Delta})/p^n$$

is crystal of locally free $\mathcal{O}_{\Delta}/p^n$-modules of rank $2g-h$. Consider the exact sequence

$$0 \to G \to X[p^n] \to H \to 0,$$

where H is a Barsotti-Tate group of height $2g-h$, induced by the embedding of G in X. This gives an exact sequence

$$\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}H,\mathcal{O}_{\Delta}) \to \mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}X[p^n],\mathcal{O}_{\Delta}) \to \mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G,\mathcal{O}_{\Delta}) \to 0.$$

Locally on $(R)_{\Delta}$, the middle term is free of rank $2g$ over $\mathcal{O}_{\Delta}/p^n$, while the left (resp. right) term is generated by $2g-h$ (resp. h) sections. Therefore, $\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}H,\mathcal{O}_{\Delta})$ and $\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G,\mathcal{O}_{\Delta})$ are free over $\mathcal{O}_{\Delta}/p^n$ of rank $2g-h$ and h. $\square$

Proposition 4.6.5. *Let R be a p-complete ring, and let G be a p-divisible group over R. The sheaf*

$$\mathcal{M}_{\Delta}(G) = \mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G,\mathcal{O}_{\Delta})$$

defined in Definition 4.2.8 is a prismatic crystal of finite locally free $\mathcal{O}_{\Delta}$-modules of rank the height of G.

Proof. Let G be a p-divisible group over R. Since $G = \operatorname{colim} G[p^n]$, we have a short exact sequence :

$$0 \to R^1\lim_n \mathcal{H}om_{(R)_{\Delta}}(u^{-1}G[p^n],\mathcal{O}_{\Delta}) \to \mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G,\mathcal{O}_{\Delta})$$
$$\to \lim_n \mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G[p^n],\mathcal{O}_{\Delta}) \to 0.$$

The first term vanishes. Indeed, let $(B,J) \in (R)_{\Delta}$ and fix some $n \geq 0$. We will show that the images of the morphisms

$$\mathrm{Hom}_{(R)_{\Delta}/(B,J)}(u^{-1}(G[p^m]),\mathcal{O}_{\Delta}) \to \mathrm{Hom}_{(R)_{\Delta}/(B,J)}(u^{-1}(G[p^n]),\mathcal{O}_{\Delta})$$

stabilize for $m \to \infty$. Let Q_m be the sequence of cokernels and set

$$M_k := \mathrm{Ext}^1_{(R)_{\Delta}/(B,J)}(u^{-1}(G[p^k]),\mathcal{O}_{\Delta})$$

for $k \geq 0$. Then there is an exact sequence

$$0 \to Q_m \to M_{m-n} \xrightarrow{p^n} M_m \to M_{p^n} \to 0.$$

Here exactness on the right follows by (locally) embedding $G[p^m]$ into (the p-completion of) an abelian scheme X and using that

$$\mathrm{Ext}^2_{(R)_{\Delta}/(B,J)}(u^{-1}(X),\mathcal{O}_{\Delta}) = 0$$

by Theorem 4.5.6. By Proposition 4.6.4 the B-module M_k is finite locally free over B/p^k for $k \geq 0$ (with rank equal to the height of G). Moreover, the canonical morphism $M_{k+1} \otimes_{B/p^{k+1}} B/p^k \to M_k$ is an isomorphism (as follows from the above exact sequence). Set

$$M := \varprojlim_k M_k.$$

Then M is a finite locally free over B (cf. [49, Tag 0D4B]) and $M \otimes_B B/p^k \cong M_k$. The exact sequence

$$0 \to Q_m \to M_{m-n} \to M_m \to M_n \to 0$$

and the snake lemma show that Q_m identifies with the cokernel of

$$M[p^m] \cong H^{-1}(M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^m) \to M[p^n] \cong H^{-1}(M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n).$$

As B is of bounded p^∞-torsion the same holds for M. This implies that for $m \to \infty$ the group $M[p^m]$, and thus Q_m, becomes constant. In the end, this implies by Mittag-Leffler the vanishing of the $R^1 \varprojlim$ term in question and thus,

$$\mathrm{Ext}^1(u^{-1}(G), \mathcal{O}_{\Delta}) \cong M$$

is finite locally free over B of rank the height of G. □

We can now summarize our discussion and prove the main result of this section.

Theorem 4.6.6. *Let R be a quasi-syntomic ring, and let G be a p-divisible group over R. The triple $\underline{\mathcal{M}}_{\Delta}(G)$ of Definition 4.2.1 is a filtered prismatic Dieudonné crystal over R.*

Proof. We have to show that $\mathcal{M}_{\Delta}(G)/\mathrm{Fil}\mathcal{M}_{\Delta}(G)$ is finite locally free over $\mathcal{O}$. The rest can then be proved exactly as in Proposition 4.5.10. First of all note that there is a natural morphism

$$\alpha\colon \mathcal{M}_{\Delta}(G) \cong \mathcal{E}xt^1_{(R)_{\mathrm{qsyn}}}(G, \mathcal{O}^{\mathrm{pris}}) \to \mathcal{E}xt^1_{(R)_{\mathrm{qsyn}}}(G, \mathcal{O}) \cong \mathrm{Lie}(G^\vee)$$

with kernel $\mathrm{Fil}\mathcal{M}_{\Delta}(G)$. As $\mathrm{Lie}(G)$ is finite locally free, it suffices to show that α is surjective. By p-completeness this may be checked after morphisms $R \to k$ with k an algebraically closed field of characteristic p. As α commutes with base change we may thus assume that $R = k$. Then the surjectivity of α follows from the comparison with the crystalline Dieudonné functor (cf. Theorem 4.3.2). □

We now state two useful properties of the prismatic Dieudonné functor : its exactness and its compatibility with Cartier duality.

Proposition 4.6.7. *Let R be a quasi-syntomic ring. The functor*

$$\mathcal{M}_{\Delta} : \mathrm{BT}(R) \to \mathrm{DM}(R), \quad G \mapsto \mathcal{M}_{\Delta}(G)$$

is exact.

Proof. Let

$$0 \to G' \to G \to G'' \to 0$$

be a short exact sequence of p-divisible groups over R, which we see as an exact sequence of abelian sheaves on $(R)_{\mathrm{qsyn}}$. Applying $R\mathcal{H}om_{(R)_{\mathrm{qsyn}}}(-, \mathcal{O}^{\mathrm{pris}})$ to it, we get a long exact sequence :

$$\mathcal{H}om_{(R)_{\mathrm{qsyn}}}(G', \mathcal{O}^{\mathrm{pris}}) \to \mathcal{M}_{\Delta}(G'') \to \mathcal{M}_{\Delta}(G) \to \mathcal{M}_{\Delta}(G') \to \mathcal{E}xt^2_{(R)_{\mathrm{qsyn}}}(G'', \mathcal{O}^{\mathrm{pris}}).$$

The first term vanishes as G' is p-divisible and $\mathcal{O}^{\mathrm{pris}}$ derived p-complete. Let us prove surjectivity of $\mathcal{M}_{\Delta}(G) \to \mathcal{M}_{\Delta}(G')$. For $n \geq 1$ consider the exact sequences

$$0 \to G'[p^n] \to G[p^n] \to H_n \to 0.$$

Then $G'' = \varinjlim_n H_n$ with injective transition maps $H_n \to H_{n+1}$ (as $G[p^n] \subseteq G' = G'[p^n]$ for all $n \geq 1$). As in the proof of Proposition 4.6.5 we can conclude that

$$\mathcal{M}_{\mathbb{A}}(G[p^n]) \to \mathcal{M}_{\mathbb{A}}(G'[p^n]),\ \mathcal{M}_{\mathbb{A}}(H_{n+1}) \to \mathcal{M}_{\mathbb{A}}(H_n)$$

are surjective. Passing to the limit of the exact sequences

$$\mathcal{M}_{\mathbb{A}}(H_n) \to \mathcal{M}_{\mathbb{A}}(G[p^n]) \to \mathcal{M}_{\mathbb{A}}(G'[p^n]) \to 0$$

implies therefore that

$$\mathcal{M}_{\mathbb{A}}(G) \to \mathcal{M}_{\mathbb{A}}(G')$$

is surjective, as desired. □

Let R be a quasi-syntomic ring and let G be a p-divisible group over R with Cartier dual G^*. Passing to the limit for the Cartier duality on finite flat group schemes yields isomorphisms

$$T_p(G^*) \cong \mathcal{H}om_R(T_pG, T_p\mu_{p^\infty}) \cong \mathcal{H}om_R(G, \mu_{p^\infty})$$

of sheaves on $(R)_{\mathrm{qsyn}}$. We first construct a canonical morphism

$$\Phi_G \colon \mathcal{M}_{\mathbb{A}}(G)^\vee \otimes_{\mathcal{O}^{\mathrm{pris}}} \mathcal{M}_{\mathbb{A}}(\mu_{p^\infty}) \to \mathcal{M}_{\mathbb{A}}(G^*),$$

where $\mathcal{M}_{\mathbb{A}}(G)^\vee$ denotes the $\mathcal{O}^{\mathrm{pris}}$-linear dual of $\mathcal{M}_{\mathbb{A}}(G)$. Recall that

$$\mathcal{M}_{\mathbb{A}}(G^*) \cong \mathcal{H}om(T_pG^*, \mathcal{O}^{\mathrm{pris}})$$

by Lemma 4.2.6. Thus we can define Φ_G by setting

$$\Phi_G(\delta \otimes l)(\alpha) := (\delta \circ \mathcal{M}_{\mathbb{A}}(\alpha))(l) \in \mathcal{O}^{\mathrm{pris}}$$

where

$$\delta \in \mathcal{M}_{\mathbb{A}}(G)^\vee, l \in \mathcal{M}_{\mathbb{A}}(\mu_{p^\infty}), \alpha \in \mathcal{H}om(G, \mu_{p^\infty}) \cong T_pG^*.$$

Clearly, the morphism Φ_G is natural in G and commutes with base change in R.

Proposition 4.6.8. *Let R be a quasi-syntomic ring. For every p-divisible group G over R, the map*

$$\Phi_G \colon \mathcal{M}_{\mathbb{A}}(G)^\vee \otimes_{\mathcal{O}^{\mathrm{pris}}} \mathcal{M}_{\mathbb{A}}(\mu_{p^\infty}) \to \mathcal{M}_{\mathbb{A}}(G^*)$$

constructed above is an isomorphism.

Proof. Both sides are locally free $\mathcal{O}^{\mathrm{pris}}$-modules of the same rank (cf. Proposition 4.6.5). Hence it suffices to see that Φ_G is surjective, which can be checked after base change $R \to k$ to perfect fields k of characteristic p. Thus, assume that $R = k$. By Theorem 4.3.2 the prismatic Dieudonné functor over k is isomorphic to the crystalline one. Let

$$\Phi_G^{\mathrm{cl}} \colon \mathcal{M}_{\mathbb{A}}(G)^\vee \otimes_{\mathcal{O}^{\mathrm{pris}}} \mathcal{M}_{\mathbb{A}}(\mu_{p^\infty}) \to \mathcal{M}_{\mathbb{A}}(G^*)$$

be the natural isomorphism coming from classical duality for the crystalline Dieudonné functor over perfect fields (cf. for example [23, Proposition 5.1.iii)]). Let

$$\Psi_{(-)} \colon \mathcal{M}_{\mathbb{A}}(-)^\vee \otimes_{\mathcal{O}^{\mathrm{pris}}} \mathcal{M}_{\mathbb{A}}(\mu_{p^\infty}) \to \mathcal{M}_{\mathbb{A}}((-)^*)$$

be any natural transformation (of functors on p-divisible groups over quasi-syntomic rings over k). Then for any morphism $\gamma \colon G \to H$ of p-divisible groups, there is an equality

$$\Psi_G(\delta \otimes l)(\alpha \circ \gamma) = \Psi_H(\delta \circ \mathcal{M}_{\mathbb{A}}(\gamma) \otimes l)(\alpha) \tag{1}$$

where $\delta \in \mathcal{M}_{\Delta}(G), l \in \mathcal{M}_{\Delta}(\mu_{p^\infty}), \alpha \in \mathcal{H}om(H, \mu_{p^\infty})$. We want to show that $\Phi_G = u\Phi_G^{\mathrm{cl}}$ for all p-divisible groups G and some unit $u \in \mathcal{O}^{\mathrm{pris}}$ (independent of G). Thus pick $\delta \in \mathcal{M}_{\Delta}(G)^\vee, l \in \mathcal{M}_{\Delta}(\mu_{p^\infty})$ and $\alpha \in \mathcal{H}om(G, \mu_{p^\infty})$. Applying (Equation (1)) to $\gamma = \alpha\colon G \to \mu_{p^\infty}$ implies

$$\Psi_G(\delta \otimes l)(\alpha) = \Psi_{\mu_{p^\infty}}(\delta \circ \mathcal{M}_{\Delta}(\alpha) \otimes l)(\mathrm{Id}_{\mu_{p^\infty}})$$

for any natural transformation $\Psi_{(-)}$ as above. In particular, Ψ (and thus $\Phi_{(-)}$ and $\Phi^{\mathrm{cl}}_{(-)}$ as examples) are determined by their behavior on $G = \mu_{p^\infty}$. For μ_{p^∞} both induce an isomorphism

$$\mathcal{M}_{\Delta}(\mu_{p^\infty})^\vee \otimes_{\mathcal{O}^{\mathrm{pris}}} \mathcal{M}_{\Delta}(\mu_{p^\infty}) \cong \mathcal{H}om(T_p(\mu_{p^\infty}), \mathcal{O}^{\mathrm{pris}}) \cong \mathcal{O}^{\mathrm{pris}}.$$

Namely, $\Phi_{\mu_{p^\infty}}$ is given by the natural evaluation, which is an isomorphism as $\mathcal{M}_{\Delta}(\mu_{p^\infty})$ is free over rank 1 (by the crystalline comparison, cf. Theorem 4.3.2). That $\Phi^{\mathrm{cl}}_{\mu_{p^\infty}}$ is an isomorphism follows from classical Dieudonné theory (cf. [23, Proposition 5.1.iii)]). Hence, $\Phi_{\mu_{p^\infty}}$ and $\Phi^{\mathrm{cl}}_{\mu_{p^\infty}}$ differ by some unit $u \in \mathcal{O}^{\mathrm{pris}}$[34]. This implies $\Phi_G = u\Phi_G^{\mathrm{cl}}$ for all G by naturality. By [23, Proposition 5.1.iii)] we can conclude. □

The main result of this text is the following theorem, whose proof will spread out over the next sections.

Theorem 4.6.9. *Let R be a quasi-syntomic ring which is flat over $\mathbb{Z}/p^n$ (for some $n > 0$) or over $\mathbb{Z}_p$. The filtered prismatic Dieudonné functor :*

$$\underline{\mathcal{M}_{\Delta}} : \mathrm{BT}(R) \to \mathrm{DF}(R)$$

is an antiequivalence between the category of p-divisible groups over R and the category of filtered prismatic Dieudonné crystals over R.

Proof. By Proposition 3.3.7 and the fact that both BT and DF are stacks on QSyn for the quasi-syntomic topology (see Proposition A.11 and Proposition 4.1.8), we can assume that moreover R is quasi-regular semiperfectoid. Then the theorem is a consequence of Theorem 4.8.5 and Theorem 4.9.5, to be proved below. □

4.7. **The prismatic Dieudonné modules of $\mathbb{Q}_p/\mathbb{Z}_p$ and μ_{p^∞}.** In this subsection, we calculate the prismatic Dieudonné crystals of $\mathbb{Q}_p/\mathbb{Z}_p$ and μ_{p^∞}. We deduce a description for all étale and multiplicative p-divisible groups. For the analogous results for the crystalline Dieudonné functor see [6, 2.2.]. Let us fix a p-complete ring R. Recall that for a p-divisible group G over R the prismatic Dieudonné crystal $\mathcal{M}_{\Delta}(G)$ is defined (cf. Definition 4.2.1) as the sheaf

$$\mathcal{M}_{\Delta}(G) := \mathcal{E}xt^1_{(R)_{\mathrm{qsyn}}}(G, \mathcal{O}^{\mathrm{pris}}) = v_*\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}(G), \mathcal{O}_{\Delta})$$

on the absolute prismatic site $(R)_{\Delta}$ of R and that

$$\mathcal{M}_{\Delta}(G) \cong \mathcal{H}om_{(R)_{\mathrm{qsyn}}}(T_pG, \mathcal{O}^{\mathrm{pris}}) = v_*\mathcal{H}om_{(R)_{\Delta}}(u^{-1}(T_pG), \mathcal{O}_{\Delta}),$$

by Lemma 4.2.6.

[34]Of course, one expects $u = \pm 1$, but as this finer statement is not necessary for us, we avoided the calculation verifying this.

Lemma 4.7.1. *The $\mathcal{O}^{\mathrm{pris}}$-module $\mathcal{M}_{\Delta}(\mathbb{Q}_p/\mathbb{Z}_p)$ is generated by the push-out of the short exact sequence*

$$0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

on $(R)_{\mathrm{qsyn}}$ along the canonical morphism $\mathbb{Z}_p \to \mathcal{O}^{\mathrm{pris}}$. More generally,

$$\mathcal{M}_{\Delta}(G) \cong \mathcal{H}om(T_p(G), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}^{\mathrm{pris}}.$$

if G is an étale p-divisible group.

Proof. This follows directly from the isomorphism

$$\mathcal{M}_{\Delta}(G) \cong \mathcal{H}om_{(R)_{\mathrm{qsyn}}}(T_pG, \mathcal{O}^{\mathrm{pris}})$$

and the fact that for an étale p-divisible group T_pG is a local system of finite free $\mathbb{Z}_p$-modules on $(R)_{\mathrm{qsyn}}$[35]. □

Let us now describe the prismatic Dieudonné crystal of μ_{p^∞}. Denote by $\widehat{\mathbb{G}_m}$ the p-adic completion of the multiplicative group scheme $\mathbb{G}_m$. First, note that

$$\mathcal{M}_{\Delta}(\mu_{p^\infty}) \cong \mathcal{E}xt^1_{(R)_{\mathrm{qsyn}}}(\widehat{\mathbb{G}_m}, \mathcal{O}^{\mathrm{pris}})$$

as $\widehat{\mathbb{G}_m}/\mu_{p^\infty}$ is uniquely p-divisible and $\mathcal{O}^{\mathrm{pris}}$ p-complete.

We cannot describe $\mathcal{M}_{\Delta}(\mu_{p^\infty})$ in general. Instead, we can describe the crystal $\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G, \mathcal{O}_{\Delta})$ on the restriction to prisms (B, J) which live over the "cyclotomic" base prism

$$(A, I) := (\mathbb{Z}_p[[q-1]], ([p]_q))$$

from Section 2.2.

The reason is that for such prisms we can use the q-logarithm from Section 2.2. Recall that

$$\mathcal{M}_{\Delta}(\mu_{p^\infty}) \cong \mathcal{H}om_{(R)_{\mathrm{qsyn}}}(\mathbb{Z}_p(1), \mathcal{O}^{\mathrm{pris}}) \cong v_*\mathcal{H}om(u^{-1}(\mathbb{Z}_p(1)), \mathcal{O}_{\Delta})$$

with $\mathbb{Z}_p(1) := T_p\mu_{p^\infty}$. In particular, the morphism

$$\log_q \colon u^{-1}(\mathbb{Z}_p(1)) \to \mathcal{O}_{\Delta}$$

from Section 2.2 defines a canonical element, which we call $\ell_q \in \mathcal{M}_{\Delta}(\mu_{p^\infty})(R)$.

Proposition 4.7.2. *Over $(A, I) = (\mathbb{Z}_p[[q-1]], ([p]_q))$, the prismatic crystal*

$$\mathcal{H}om_{(R)_{\Delta}}(u^{-1}(\mathbb{Z}_p(1)), \mathcal{O}_{\Delta})$$

is free of rank 1, generated by ℓ_q. Moreover, the Frobenius on $\mathcal{H}om(u^{-1}(\mathbb{Z}_p(1)), \mathcal{O}_{\Delta})$ sends ℓ_q to $[p]_q\ell_q$.

Proof. Let (B, J) be a prism over (A, I). It suffices to show that

$$\mathrm{Ext}^1(u^{-1}(\widehat{\mathbb{G}_m})_{|(B,J)}, \mathcal{O}_{\Delta}),$$

where we mean Ext^1 in the category of abelian sheaves on the site of prisms over (B, J), is freely generated by ℓ_q. By Proposition 4.6.5 this group satisfies base change in (B, J). From the case $(B, J) = (A, I)$, a comparison with q-de Rham cohomology (cf. [12, Theorem 16.17]) and the spectral sequence from Section 4.4 one can conclude that it is free of rank 1 over B. To show that ℓ_q is a generator one may pass to the case that $(B, J) = (W(k), (p))$ for k an algebraically closed field

[35]Here, we did some abuse of notation and denoted by $\mathbb{Z}_p$ the sheaf $S \mapsto \mathrm{Hom}_{\mathrm{cts}}(\pi_0(S), \mathbb{Z}_p)$ on $(R)_{\mathrm{qsyn}}$, which is usually called $\underline{\mathbb{Z}_p}$.

of characteristic p. Then the comparison with the crystalline Dieudonné crystal (cf. Theorem 4.3.2) reduces to an analogous statement for the usual logarithm as for $q = 1$ the q-logarithm becomes the logarithm. Let R be a general ring of characteristic p and let $R' \to R$ be a surjection of schemes with a PD-structure $\{\gamma_n\}_{n\geq 0}$ on $K := \ker(R' \to R)$ and assume p nilpotent in R'. Then there is the canonical morphism

$$\log\colon \mathbb{Z}_p(1)(R) \to R',\ x \mapsto \log([x])$$

where $[-]\colon \varprojlim_{x\mapsto x^p} R \to R'$ is the Teichmüller lift and log the crystalline logarithm

$$\log\colon 1 + K \to R',\ y \mapsto \sum_{n=1}^{\infty} (-1)^{n-1}(n-1)!\gamma_n(y-1)$$

(which makes sense as $[x] \in 1+K$). But it is known that the logarithm generates the crystalline Dieudonné crystal of μ_{p^∞} (cf. [6, 2.2.3.Corollaire]). Finally the action of Frobenius on ℓ_q can be calculated using Lemma 2.2.2:

$$\varphi_{\mathcal{H}om(u^{-1}(\mathbb{Z}_p(1)),\mathcal{O}_{\Delta})}(\ell_q)(x) = \frac{q^p-1}{\log(q)}\log(x^p) = \frac{q^p-1}{q-1}\ell_q(x) = [p]_q\ell_q(x)$$

for $x \in \mathbb{Z}_p(1)$. □

Remark 4.7.3. Note that, when $pR = 0$, the identification between the prismatic and crystalline Dieudonné modules from Theorem 4.3.2 is linear *over the isomorphism* $\Delta_R \cong A_{\mathrm{crys}}(R)$ from Lemma 3.4.3. This explains why the map $x \mapsto \log_q([x^{1/p}]_{\tilde{\theta}})$ is sent to $x \mapsto \log([x])$ (and not something like $x \mapsto \log([x^{1/p}])$, which would not make sense as $[x^{1/p}] - 1$ need not have divided powers), cf. the remark after Lemma 3.4.3.

Assume now that R is an $A/I = \mathbb{Z}[\zeta_p]$-algebra.

Corollary 4.7.4. *Let G be a multiplicative p-divisible group over R. Then there is a canonical isomorphism*

$$u^{-1}(\mathcal{H}om(G,\mu_{p^\infty})) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\Delta} \cong \mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G,\mathcal{O}_{\Delta})_{|(R/A)_{\Delta}}$$

induced by sending $f\colon G \to \mu_{p^\infty}$ to the evaluation of the morphism induced by f:

$$\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}\mu_{p^\infty},\mathcal{O}_{\Delta})_{|(R/A)_{\Delta}} \to \mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G,\mathcal{O}_{\Delta})_{|(R/A)_{\Delta}}$$

on ℓ_q.

Proof. The morphism (and the claim that it is an isomorphism) commutes with étale localisation on R. In particular, we may assume that $G \cong \mu_{p^\infty}^d$. Then the claim follows from Proposition 4.7.2 and additivity of the right hand side. □

The important corollary of these computations is a description of the action of the prismatic Dieudonné functor on morphisms $\mathbb{Q}_p/\mathbb{Z}_p \to \mu_{p^\infty}$. Set

$$\mathbb{Z}_p^{\mathrm{cycl}} := (\varinjlim_n \mathbb{Z}_p[\zeta_{p^n}])^{\wedge}_p.$$

As usual we get the elements $\varepsilon = (1, \zeta_p, \ldots)$, $q := [\varepsilon] \in A_{\inf}(\mathbb{Z}_p^{\mathrm{cycl}})$ and $\tilde{\xi} := \frac{q^p-1}{q-1}$.

Lemma 4.7.5. *Let R be a quasi-regular semiperfectoid ring over $\mathbb{Z}_p^{\mathrm{cycl}}$. Then the morphism*

$$\mathbb{Z}_p(1)(R) \cong \mathrm{Hom}_R(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}) \xrightarrow{M_{\Delta}(-)} \mathrm{Hom}_{\mathrm{DM}(R)}(M_{\Delta}(\mu_{p^\infty})), M_{\Delta}(\mathbb{Q}_p/\mathbb{Z}_p)) \cong \Delta_R^{\varphi=\tilde{\xi}}$$

is given the map which sends $x \in \mathbb{Z}_p(1)(R)$ to $\log_q([x^{1/p}]_{\tilde{\theta}}) \in \Delta_R^{\varphi=\tilde{\xi}}$.

Proof. First note, that

$$\mathrm{Hom}_{\mathrm{DM}(R)}(M_{\Delta}(\mu_{p^\infty})), M_{\Delta}(\mathbb{Q}_p/\mathbb{Z}_p)) \cong \Delta_R^{\varphi=\tilde{\xi}}$$

by evaluating a homomorphism $M_{\Delta}(\mu_{p^\infty}) \to M_{\Delta}(\mathbb{Q}_p/\mathbb{Z}_p) \cong \Delta_R$ on ℓ_q. The identification of $M_{\Delta}(-)$ on a homomorphism $f\colon \mathbb{Q}_p/\mathbb{Z}_p \to \mu_{p^\infty}$ follows easily from the natural isomorphism

$$M_{\Delta}(G) \cong \mathrm{Hom}_{(R)_{\Delta}}(u^{-1}(T_p(G)), \mathcal{O}_{\Delta})$$

for a p-divisible group G over R and Proposition 4.7.2, Lemma 4.7.1. $\square$

The following theorem, proved in the companion paper [1], is crucial and, unfortunately, relies (in its full generality) on a deep result in algebraic K-theory.[36]

Theorem 4.7.6. *Let R be quasi-regular semiperfectoid. The prismatic Dieudonné functor induces an isomorphism*

$$\mathrm{Hom}_R(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}) \cong \mathrm{Hom}_{\mathrm{DM}(R)}(M_{\Delta}(\mu_{p^\infty}), M_{\Delta_R}(\mathbb{Q}_p/\mathbb{Z}_p)).$$

Proof. Both sides satisfy quasi-syntomic descent in R. For the left hand side this follows from Proposition A.11. For the right hand side, this was proven in Proposition 4.1.8. Thus we may assume R is a $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. By Lemma 4.7.5 we have to prove bijectivity of the q-logarithm

$$\log_q\colon \mathbb{Z}_p(1)(R) \to \Delta_R^{\varphi=\tilde{\xi}}.$$

Let $\hat{\Delta}_R$ be the Nygaard completion on Δ_R. By Theorem 3.4.5 and the remark following it, $\hat{\Delta}_R^{\varphi=\tilde{\xi}}$ identifies with the second p-completed topological cylic homology group $\pi_2(\mathrm{TC}(R;\mathbb{Z}_p))$ of R. By [1, Corollary 1.5], we know that the composition

$$\mathbb{Z}_p(1)(R) \xrightarrow{\log_q} \Delta_R^{\varphi=\tilde{\xi}} \to \hat{\Delta}_R^{\varphi=\tilde{\xi}}$$

is a bijection because it identifies with minus the cyclotomic trace to which one can apply the results of [18]. Hence, it suffices to show that the morphism

$$\pi\colon \Delta_R^{\varphi=\tilde{\xi}} \to \hat{\Delta}_R^{\varphi=\tilde{\xi}}$$

is injective. By definition of the Nygaard filtration the Frobenius on Δ_R factors through a map

$$\psi\colon \hat{\Delta}_R \to \Delta_R,$$

i.e., $\varphi = \psi \circ \pi$. Let $x \in \Delta_R^{\varphi=\tilde{\xi}}$ be in the kernel of π. Then

$$0 = \psi(\pi(x)) = \varphi(x) = \tilde{\xi}x,$$

which implies $x = 0$ as $\tilde{\xi}$ is a non-zero divisor in Δ_R. This finishes the proof. $\square$

[36] If R is regular semiperfect, the result is proven more elementary in [48]. If $p \neq 2$, then the case of p-torsion free regular semiperfectoid rings can be deduced from this.

In the case when R is of characteristic p, one can directly see that

$$A_{\mathrm{crys}}(R)^{\varphi=p^i} \cong \mathbb{\Delta}_R^{\varphi=\tilde{\xi}^i} \cong \hat{\mathbb{\Delta}}_R^{\varphi=\tilde{\xi}^i} \cong \hat{A}_{\mathrm{crys}}^{\varphi=p^i},$$

cf. [11, Proposition 8.18].

4.8. **Fully faithfulness for p-divisible groups.** In this section we want to prove fully faithfulness of the prismatic Dieudonné functor for p-divisible groups over (certain) quasi-syntomic rings. We will do this by descent from the quasi-regular semiperfectoid case.

We start with a preliminary technical result (Proposition 4.8.1). Fix for all this subsection a complete algebraically closed extension C of $\mathbb{Q}_p$.

Let R be a quasi-regular semiperfectoid ring (for the moment we do not assume that R is an $\mathcal{O}_C$-algebra). Let G be a p-divisible group over R. The formal scheme

$$T_pG \cong \mathrm{Spf}(R_G)$$

is represented by an R-algebra R_G which is again quasi-regular semiperfectoid. Indeed, as finite locally free group schemes are quasi-syntomic over R the ring R_G is quasi-syntomic. Moreover, R_G/p is semiperfect by [48, Chapter 4.3]. The Yoneda lemma shows that

$$\mathbb{\Delta}_{R_G} \cong \mathrm{Hom}_{(R)_{\mathrm{qsyn}},\mathrm{Sets}}(T_pG, \mathcal{O}^{\mathrm{pris}})$$

where the right-hand side denotes natural transformations of set-valued sheaves on the quasi-syntomic site $(R)_{\mathrm{qsyn}}$ of R. By Lemma 4.2.6

$$M_{\mathbb{\Delta}}(G) \cong \mathrm{Hom}_{(R)_{\mathrm{qsyn}}}(T_pG, \mathcal{O}^{\mathrm{pris}})$$

where the right-hand side denotes (as before) natural transformations of *abelian* sheaves. In particular, there exists a canonical morphism

$$M_{\mathbb{\Delta}}(G) \to \mathbb{\Delta}_{R_G}.$$

which is injective.

Under strong assumptions on R, one can prove more about this map. Let us fix as usual a compatible system $\varepsilon \in \mathcal{O}_C^\flat$ of primitive p^n-th roots of unity and let $\mu = [\varepsilon] - 1$.

Proposition 4.8.1. *Let R be a p-torsion free perfectoid $\mathcal{O}_C$-algebra which is integrally closed in $R[1/p]$ and let G be a p-divisible group over R such that*

$$G \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R[1/p]) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^h.$$

Then the cokernel of the natural morphism

$$\mathbb{\Delta}_{R_G}^* \to M_{\mathbb{\Delta}}(G)^*,$$

where $(-)^*$ refers to the $\mathbb{\Delta}_R$-linear dual, is killed by μ.

Remark 4.8.2. As noticed in [48], and as will be apparent in the proof of Theorem 4.8.5 below, one should think to this statement as saying that, for any p-divisible group G over a big ring R as in the proposition, there are *many* morphisms from $\mathbb{Q}_p/\mathbb{Z}_p$ to G.

Proof. Fix an isomorphism

$$\gamma\colon (\mathbb{Q}_p/\mathbb{Z}_p)^h \to G \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R[1/p]).$$

The choice of ε defines an isomorphism of $\mathbb{Q}_p/\mathbb{Z}_p \cong \mu_{p^\infty}$ over $\mathrm{Spec}(R[1/p])$. Dualising γ yields therefore an isomorphism

$$\eta\colon (\mathbb{Q}_p/\mathbb{Z}_p)^h \to G^\vee \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R[1/p])$$

where $G^\vee$ is the Cartier dual of G. As R is integrally closed in $R[1/p]$ the morphisms γ and η extend to morphism

$$\tilde{\gamma}\colon (\mathbb{Q}_p/\mathbb{Z}_p)^h \to G$$

and

$$\tilde{\eta}\colon (\mathbb{Q}_p/\mathbb{Z}_p)^h \to G^\vee$$

over R. The composition

$$(\mathbb{Q}_p/\mathbb{Z}_p)^h \xrightarrow{\tilde{\gamma}} G \xrightarrow{\tilde{\eta}^\vee} \mu_{p^\infty}^h$$

is given by the diagonal morphism induced by ε. Indeed, by construction this holds over $R[1/p]$. But as R is p-torsion free the functor

$$G \mapsto G \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R[1/p])$$

is faithful, which implies the claim over R. Next we claim that the cokernel of

$$\Delta^*_{R_G} \to M_{\Delta}(G)^*$$

is μ-torsion. For this consider the diagram

$$\begin{tikzcd}
\Delta^*_{R_{(\mathbb{Q}_p/\mathbb{Z}_p)^h}} \arrow[r] \arrow[d] & M_{\Delta}((\mathbb{Q}_p/\mathbb{Z}_p)^h)^* \arrow[d, "f"] \\
\Delta^*_{R_G} \arrow[r] \arrow[d] & M_{\Delta}(G)^* \arrow[d, "g"] \\
\Delta^*_{R_{\mu^h_{p^\infty}}} \arrow[r] & M_{\Delta}(\mu^h_{p^\infty})^*
\end{tikzcd}$$

with f, g induced by $\tilde{\gamma}$ resp. $\tilde{\eta}$. The composition $g \circ f$ is given, using the description of $M_{\Delta}(\mathbb{Q}_p/\mathbb{Z}_p)$ and $M_{\Delta}(\mu_{p^\infty})$ from Section 4.7, by multiplication with μ (as follows from Lemma 4.7.5). Pick $x \in M_{\Delta}(G)^*$. Then

$$g(\mu x) = g(f(y))$$

for some $y \in M_{\Delta}((\mathbb{Q}_p/\mathbb{Z}_p)^h)$. But g is injective, as $f \circ g$ is an isomorphism after inverting μ, Δ_R is μ-torsion free (because R is perfectoid and p-torsion free) and $M_{\Delta}(G)^*$ is of rank h. Thus

$$\mu x = f(y).$$

Using Lemma 4.8.3 and the above diagram we can conclude that μx lies in the image of

$$\Delta^*_{R_G} \to M_{\Delta}(G)^*$$

as claimed. $\square$

Lemma 4.8.3. *In Proposition 4.8.1, assume $G = \mathbb{Q}_p/\mathbb{Z}_p$. Then the natural morphism*

$$\Delta^*_{R_G} \to M_{\Delta}(G)^*$$

is surjective.

Proof. In this case,

$$\Delta_{R_G}$$

is given by the ring of continuous functions $\mathbb{Z}_p \to \Delta_R$ where Δ_R is given the discrete topology, while $M_{\Delta}(G)$ embeds into Δ_{R_G} as the Δ_R-module of constant functions. Evaluating at some point of $\mathbb{Z}_p$ defines a linear form which maps to a generator of $M_{\Delta}(G)^*$. □

We will need the following result.

Lemma 4.8.4. *Let (C, J) be an henselian pair and let $\overline{G}$ be a p-divisible group over C/J. Then there exists a p-divisible group G over C such that*

$$G \otimes_C C/J \cong \overline{G}.$$

Proof. Set h as the height of $\overline{G}$. Let BT^h_n be the Artin stack (over $\mathrm{Spec}(\mathbb{Z})$) of n-truncated Barsotti-Tate groups of height h. Then for any $n \geq 1$ the morphism

$$\mathrm{BT}^h_n \to \mathrm{BT}^h_{n-1}$$

is a smooth morphism between smooth Artin stacks (cf. [32, Section 2] resp. [25, Thm 4.4.]). By [21, Theorem, page 568] (which extends to the non-noetherian case by passing to the limit) any section $D \to C/J$ of some smooth C-algebra D extends to a section $D \to C$. These statements imply that inductively, we can lift $\overline{G}[p^n]$ to a truncated p-divisible group H_n over C. Then finally

$$G := \varinjlim_n H_n$$

yields the desired lift over $\overline{G}$. □

The main result of this subsection is the following.

Theorem 4.8.5. *If R is a quasi-regular semiperfectoid ring, flat over $\mathbb{Z}/p^n$ (for some $n > 0$) or $\mathbb{Z}_p$, the prismatic Dieudonne functor over R is fully faithful for p-divisible groups.*

We point out that this is statement is for the prismatic Dieudonné functor, and not only for the *filtered* prismatic Dieudonné functor.

Proof. From Theorem 4.7.6 we know that the prismatic Dieudonné functor induces an isomorphism

$$\mathrm{Hom}_R(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}) \xrightarrow{\cong} \mathrm{Hom}_{\Delta_R}(M_{\Delta}(\mu_{p^\infty}), M_{\Delta}(\mathbb{Q}_p/\mathbb{Z}_p)),$$

i.e., that it is fully faithful for morphisms $\mathbb{Q}_p/\mathbb{Z}_p \to \mu_{p^\infty}$ over R. We want to deduce that it is fully faithful in general. For this, we follow the strategy of [48, Section 4.4.].

Let G_1, G_2 be two p-divisible groups over R. The R-algebra

$$R_{G_1,G_2} = R_{G_1} \hat{\otimes}_R R_{\check{G}_2}$$

represents $T_pG_1 \times_{\mathrm{Spf}(R)} T_p\check{G}_2$ and is again quasi-regular semiperfectoid. Over R_{G_1,G_2}, there are universal morphisms $\mathbb{Q}_p/\mathbb{Z}_p \to G_{1,R_{G_1,G_2}}$ and $G_{2,R_{G_1,G_2}} \to \mu_{p^\infty}$

(we denote by $G_{1,R_{G_1,G_2}}$ and $G_{2,R_{G_1,G_2}}$ the base changes of G_1 and G_2 to R_{G_1,G_2}). Passing to prismatic Dieudonné modules, we get natural morphisms

$$M_{\mathbb{\Delta}}(G_{1,R_{G_1,G_2}}) \to M_{\mathbb{\Delta}}(\mathbb{Q}_p/\mathbb{Z}_p)\ ,\ M_{\mathbb{\Delta}}(\mu_{p^\infty}) \to M_{\mathbb{\Delta}}(G_{2,R_{G_1,G_2}}).$$

Let $f\colon M_{\mathbb{\Delta}}(G_2) \to M_{\mathbb{\Delta}}(G_1)$ be a morphism of prismatic Dieudonné modules. Base changing f to R_{G_1,G_2}, we get a morphism $M_{\mathbb{\Delta}}(G_{2,R_{G_1,G_2}}) \to M_{\mathbb{\Delta}}(G_{1,R_{G_1,G_2}})$, which we can pre- and post-compose with the canonical morphisms

$$M_{\mathbb{\Delta}}(G_{1,R_{G_1,G_2}}) \to M_{\mathbb{\Delta}}(\mathbb{Q}_p/\mathbb{Z}_p) \quad , \quad M_{\mathbb{\Delta}}(\mu_{p^\infty}) \to M_{\mathbb{\Delta}}(G_{2,R_{G_1,G_2}})$$

to get a morphism

$$\beta_f : M_{\mathbb{\Delta}}(\mu_{p^\infty}) \to M_{\mathbb{\Delta}}(\mathbb{Q}_p/\mathbb{Z}_p)$$

of prismatic Dieudonné modules over R_{G_1,G_2}. By Theorem 4.7.6, it comes from a morphism

$$\eta_f \in \mathrm{Hom}_{R_{G_1,G_2}}(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}).$$

The morphism η_f is the same thing as a family $(s_n)_{n\geq 0}$ of elements in R_{G_1,G_2}, with $s_0 = 1$, $s_{n+1}^p = s_n$ for all n. The same arguments as in [48, Chapter 4.4.] (which require Proposition 4.6.8) prove that s_n corresponds (by the very definition of R_{G_1,G_2}) to a morphism of finite locally free group schemes $G_1[p^n] \to G_2[p^n]$. The condition $s_{n+1}^p = s_n$ for all n means that these morphisms combine to a morphism

$$\alpha_R(f) : G_1 \to G_2$$

of p-divisible groups over R. In other words, we have constructed a map[37]

$$\alpha_R\colon \mathrm{Hom}_{\mathbb{\Delta}_R}(M_{\mathbb{\Delta}}(G_2), M_{\mathbb{\Delta}}(G_1)) \to \mathrm{Hom}_R(G_1, G_2),$$

which is moreover natural in R and a retraction of

$$\mathrm{Hom}_R(G_1, G_2) \xrightarrow{M_{\mathbb{\Delta}}(-)} \mathrm{Hom}_{\mathbb{\Delta}_R}(M_{\mathbb{\Delta}}(G_2), M_{\mathbb{\Delta}}(G_1)).$$

To prove the theorem, it therefore suffices to show that

$$\alpha_R\colon \mathrm{Hom}_{\mathbb{\Delta}_R}(M_{\mathbb{\Delta}}(G_2), M_{\mathbb{\Delta}}(G_1)) \to \mathrm{Hom}_R(G_1, G_2),$$

is injective, for any p-divisible groups G_1, G_2 over R. This is the statement we will prove, using the assumption that R is flat over $\mathbb{Z}/p^n$, for some $n > 0$, or over $\mathbb{Z}_p$, which was not used yet. To shorten notation, we simply say in the rest of the proof that R is flat over $\mathbb{Z}/p^n$, allowing the limit case $n = \infty$, which corresponds to the case where R is flat over $\mathbb{Z}_p$ (i.e., p-torsion free).

As one can argue quasi-syntomic locally, we can replace R by its base change to $\mathcal{O}_C/p^n$, which we still denote by R. By Proposition 3.6.2 the prism $\mathbb{\Delta}_R$ of R is μ-torsion free. This will play an important role. We start by some observations.

(1) There exists a p-torsion free perfectoid $\mathcal{O}_C$-algebra S mapping surjectively onto R. Moreover one can assume that S is henselian along $\ker(S \to R)$. Indeed, since R lives over $\mathcal{O}_C/p^n$, we have in particular a map $\mathcal{O}_C \to R$. It extends to a ring map :

$$\mathcal{O}_C\langle X_i^{1/p^\infty}, i \in R^\flat\rangle \to R, \quad x_i^{1/p^n} \mapsto (i^\sharp)^{1/p^n}.$$

By completeness, it is enough to check surjectivity modulo p and this holds true by semiperfectness of R/p. By Corollary 2.1.10, we can assume that S is henselian along $\ker(S \to R)$.

[37] This map of course depends on G_1 and G_2, although the notation does not indicate it.

From now on, we fix such a ring S.

(2) Any p-divisible group G over R lifts to a p-divisible group G_S over S. This is the content of Lemma 4.8.4.

(3) Let G_1, G_2 be two p-divisible groups over R. Let R' be a quasi-regular semiperfectoid ring to which R maps, such that the induced map $\Delta_R \to \Delta_{R'}$ is injective. Set

$$G_1' := G_1 \otimes_R R' \ , \ G_2' := G_2 \otimes_R R'.$$

As $M_{\Delta}(G_1), M_{\Delta}(G_2)$ (resp. $M_{\Delta}(G_1'), M_{\Delta}(G_2')$) are finite locally free over Δ_R (resp. over $\Delta_{R'}$), the map

$$\mathrm{Hom}_{\Delta_R}(M_{\Delta}(G_2), M_{\Delta}(G_1)) \to \mathrm{Hom}_{\Delta_{R'}}(M_{\Delta}(G_2'), M_{\Delta}(G_1'))$$

is injective as well, i.e., the base change functor

$$\mathrm{DM}(R) \to \mathrm{DM}(R')$$

is faithful. By naturality of α we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\Delta_R}(M_{\Delta}(G_2), M_{\Delta}(G_1)) \arrow[r, "\alpha_R"] \arrow[d] & \mathrm{Hom}_R(G_1, G_2) \arrow[d] \\ \mathrm{Hom}_{\Delta_{R'}}(M_{\Delta}(G_2'), M_{\Delta}(G_1')) \arrow[r, "\alpha_{R'}"] & \mathrm{Hom}_{R'}(G_1', G_2'). \end{tikzcd}$$

Let $f \in \mathrm{Hom}_{\Delta_R}(M_{\Delta}(G_2), M_{\Delta}(G_1))$, such that $\alpha_R(f) = 0$. Assume that one can prove that the kernel of $\alpha_{R'}$ is μ-torsion. Then if f' denotes the base change of f to $\Delta_{R'}$, the previous diagram and the assumption show that $\mu.f = 0$. But by Proposition 3.6.2 (and the assumption made before that R is the base change to $\mathcal{O}_C/p^n$ of a ring flat over $\mathbb{Z}/p^n$), Δ_R is μ-torsion free, and so also is the finite projective Δ_R-module $M_{\Delta}(G_1)$. Thus $f = 0$.

Therefore, to prove fully faithfulness over R, it is enough, for each $G_1, G_2 \in \mathrm{BT}(R)$, to find a map $R \to R'$, with R' quasi-regular semiperfectoid, such that $\Delta_R \to \Delta_{R'}$ is injective and such that the kernel of the map $\alpha_{R'}$ (attached to the base change of G_1, G_2 to R') is μ-torsion.

(4) Let G be any p-divisible group over R. By Observation (2), G lifts to a p-divisible group G_S over S. Let S' be the integral closure of S in the $S[1/p]$-algebra parametrizing trivializations of G_S over $S[1/p]$. By the almost purity theorem, [12, Theorem 10.8], S' is a perfectoid ring and is the p-completion of a filtered colimit of almost finite étale extensions of S ; in particular, the map $S \to S'$ is p-completely almost faithfully flat. Set $R' = R\hat{\otimes}_S S'$. We claim that the map

$$\Delta_R \to \Delta_{R'}$$

is injective. This map is the $(p, \tilde{\xi})$-completed base change of the map $\Delta_S \to \Delta_{S'}$, which is $(p, \tilde{\xi})$-completely almost faithfully flat, and therefore is itself $(p, \tilde{\xi})$-completely almost faithfully flat, and therefore almost injective, i.e., every element in $\ker(\Delta_R \to \Delta_{R'})$ is killed by $W(\mathfrak{m}^\flat)$, where $\mathfrak{m}^\flat \subseteq \mathcal{O}_C^\flat$ is the maximal ideal. As we have already seen, Δ_R has no μ-torsion and so the kernel is zero on the nose, as $\mu \in W(\mathfrak{m}^\flat)$.

We know fix G_1 and G_2 two p-divisible groups over R. As shown by Observation (3), it is enough to find a quasi-regular semiperfectoid ring R' over R, such that Δ_R injects in $\Delta_{R'}$ and such that the kernel of $\alpha_{R'}$ is killed by μ. We take $R' = S'\hat{\otimes}_S R$, where S' is defined as in Observation (4) for the choice of a lift of the p-divisible group $G = \check{G}_2$ to S. By the same observation, the map $\Delta_R \to \Delta_{R'}$ is injective. It remains to check the assertion on the kernel of $\alpha_{R'}$.

From now on and until the end of this proof, to keep the notations light, we rename S' as S and R' as R. As a consequence of the definitions, the quasi-regular semiperfectoid ring R is the quotient of a perfectoid $\mathcal{O}_C$-algebra S, which is integrally closed in $S[1/p]$ and such that $\check{G}_{S,2}$ is trivialized on $S[1/p]$. By Proposition 4.8.1, the natural map

$$\Delta^*_{S_{\check{G}_{S,2}}} \to M_{\Delta}(\check{G}_{S,2})^*$$

has its cokernel killed by μ. By base change along $\Delta_S \to \Delta_R$, we get that the cokernel of the map

$$\mathrm{Hom}_{\Delta_R}(\Delta_{S_{\check{G}_{S,2}}}\hat{\otimes}_{\Delta_S}\Delta_R, \Delta_R) \to M_{\Delta}(\check{G}_{S,2})^*\hat{\otimes}_{\Delta_S}\Delta_R = M_{\Delta}(\check{G}_2)^*$$

(the last equality comes from the fact that $G_{S,2} \otimes_S R = G_2$ by definition and the fact that the prismatic Dieudonné module is finite locally free). Since

$$S_{\check{G}_{S,2}}\hat{\otimes}_S R = R_{\check{G}_2},$$

by the Künneth formula for prismatic cohomology (Proposition 3.5.1) and p-complete flatness of $S \to S_{\check{G}_{S,2}}$, we have

$$\Delta_{S_{\check{G}_{S,2}}}\hat{\otimes}_{\Delta_S}\Delta_R = \Delta_{R_{\check{G}_2}}$$

and so we deduce that the map

$$\Delta^*_{R_{\check{G}_2}} \to M_{\Delta}(\check{G}_2)^* \tag{2}$$

has cokernel killed by μ.

Let $f \in \mathrm{Hom}_{\mathrm{DM}(R)}(M_{\Delta}(G_2), M_{\Delta}(G_1))$. We explained at the beginning of the proof that the associated morphism $\alpha_R(f)$ is the same thing as an element

$$\eta_f \in \mathrm{Hom}_{R_{G_1,G_2}}(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty})$$

and we want to prove that if η_f is zero, then $f = 0$. By definition, η_f is the morphism corresponding to the composition

$$M_{\Delta}((\mu_{p^\infty})_{R_{G_1,G_2}}) \to M_{\Delta}(G_{2,R_{G_1,G_2}}) \xrightarrow{f_{R_{G_1,G_2}}} M_{\Delta}(G_{1,R_{G_1,G_2}}) \to M_{\Delta}((\mathbb{Q}_p/\mathbb{Z}_p)_{R_{G_1,G_2}}),$$

where the left and right morphisms are induced by the two universal morphisms coming from the definition of R_{G_1,G_2}.

Assume that $\eta_f = 0$. To conclude the proof of the theorem, it suffices to prove that this implies that $f = 0$. Applying the considerations preceeding Proposition 4.8.1 to the quasi-regular semiperfectoid ring $R_{\check{G}_2}$ and to the p-divisible group $G_{1,R_{\check{G}_2}}$, we know that the natural map

$$M_{\Delta}(G_{1,R_{\check{G}_2}}) \to \Delta_{R_{G_1,G_2}}$$

is injective, but by construction this map identifies with the map

$$M_{\Delta}(G_{1,R_{\check{G}_2}}) \to M_{\Delta}(G_{1,R_{G_1,G_2}}) \to M_{\Delta}((\mathbb{Q}_p/\mathbb{Z}_p)_{R_{G_1,G_2}}),$$

where the two maps are the natural ones. Considering the diagram

$$\begin{tikzcd}
M_{\Delta}((\mu_{p^\infty})_{R_{\check{G}_2}}) \arrow[r] \arrow[d] & M_{\Delta}((\mu_{p^\infty})_{R_{G_1,G_2}}) \arrow[d] \arrow[ddd, bend left, "\eta_f"] \\
M_{\Delta}(G_{2,R_{\check{G}_2}}) \arrow[r] \arrow[d, "f_{R_{\check{G}_2}}"] & M_{\Delta}(G_{2,R_{G_1,G_2}}) \arrow[d, "f_{R_{G_1,G_2}}"] \\
M_{\Delta}(G_{1,R_{\check{G}_2}}) \arrow[r] & M_{\Delta}(G_{1,R_{G_1,G_2}}) \arrow[d] \\
 & M_{\Delta}((\mathbb{Q}_p/\mathbb{Z}_p)_{R_{G_1,G_2}}).
\end{tikzcd}$$

we deduce that the composition

$$M_{\Delta}((\mu_{p^\infty})_{R_{\check{G}_2}}) \to M_{\Delta}(G_{2,R_{\check{G}_2}}) \xrightarrow{f_{R_{\check{G}_2}}} M_{\Delta}(G_{1,R_{\check{G}_2}})$$

is the zero map. Let α be the image of the generator of $M_{\Delta}((\mu_{p^\infty})_{R_{\check{G}_2}})$ in

$$M_{\Delta}(G_{2,R_{\check{G}_2}}) = M_{\Delta}(G_2) \otimes_{\Delta_R} \Delta_{R_{\check{G}_2}}.$$

Let $v \in M_{\Delta}(G_2)$. Since the cokernel of (2) is killed by μ, there exists $\lambda \in \Delta^*_{R_{\check{G}_2}}$ such that

$$\lambda(\alpha) = \mu v.$$

Therefore, considering the commutative diagram

$$\begin{tikzcd}
 & M_{\Delta}((\mu_{p^\infty})_{R_{\check{G}_2}}) \arrow[d] \\
M_{\Delta}(G_2) \arrow[r] \arrow[d, "f"] & M_{\Delta}(G_{2,R_{\check{G}_2}}) \arrow[d, "f_{R_{G_1,G_2}}"] \\
M_{\Delta}(G_1) \arrow[r] & M_{\Delta}(G_{1,R_{\check{G}_2}}).
\end{tikzcd}$$

we deduce that

$$\mu f(v) = f(\mu v) = f(\lambda(\alpha)) = 0,$$

since the composition

$$M_{\Delta}((\mu_{p^\infty})_{R_{\check{G}_2}}) \to M_{\Delta}(G_{2,R_{\check{G}_2}}) \xrightarrow{f_{R_{\check{G}_2}}} M_{\Delta}(G_{1,R_{\check{G}_2}})$$

has been proved to be zero. Hence, $\mu.f = 0$, which ends the proof. $\square$

Remark 4.8.6. For quasi-regular semiperfect rings (the case $n = 1$, for which the flatness condition is empty), one could instead use the results of [48, §4.3]. However, the proof of loc. cit. is more involved and requires results on the Hodge-Tate sequence.

Remark 4.8.7. Let R be a quasi-regular semiperfectoid algebra, flat over $\mathcal{O}_C/p^n$ (including the limit case $n=\infty$, i.e. the case where R is flat over $\mathcal{O}_C$). The ring R/p is quasi-regular semiperfect, so it would be tempting to try to prove the theorem by reduction to the characteristic p case. For this, it is enough (by Observation (3) in the above proof) to prove that the natural morphism

$$\Delta_R \to \Delta_{R/p}$$

is injective. This works fine if $n=\infty$, i.e. if R is p-torsion free. This can also be checked by hand in the special case $R=\mathcal{O}_C/p^n$. One could then try to use that the morphism

$$\Delta_R \to \Delta_{R/p}$$

is the completed base change along the map $\Delta_{\mathcal{O}_C/p^n} \to \Delta_R$ of the injective map

$$\Delta_{\mathcal{O}_C/p^n} \to \Delta_{\mathcal{O}_C/p}$$

and to prove that Δ_R is (p,I)-completely flat over $\Delta_{\mathcal{O}_C/p^n}$. But one issue is that in general, the completed base change along a f-completely flat morphism between f-complete rings with no f-torsion need not preserve short exact sequences of derived f-adically complete modules. Here is an explicit counterexample. Set

$$R := \mathbb{Z}[f,x]^{\wedge_f}$$

and

$$R' := \mathbb{Z}[f,x^{\pm 1}]^{\wedge_f}.$$

Then the morphism $R \to R'$ is f-completely flat, and even flat. Consider the short exact sequence

$$0 \to \widehat{\bigoplus_{i\geq 0}} Rs_i \xrightarrow{\alpha} \widehat{\bigoplus_{i\geq 0}} Rt_i \to Q \to 0$$

where $\alpha(s_i) := ft_i - xt_{i-1}$ (with $t_{-1} := 0$).[38] Let $q_i \in Q$ be the image of t_i. By construction

$$fq_i = xq_{i-1}$$

for $i \geq 0$ and Q is derived f-complete (cf. [49, Example 09AT]). We claim that the sequence

$$0 \to \widehat{\bigoplus_{i\geq 0}} R's_i \xrightarrow{\alpha} \widehat{\bigoplus_{i\geq 0}} R't_i \to H^0(Q\widehat{\otimes}_R R') \to 0$$

is not exact. Indeed,

$$H^{-1}(Q\widehat{\otimes}_R R') \cong T_f(Q\otimes_R R') \neq 0$$

as the element

$$(q_0\otimes 1, q_1\otimes 1/x, q_2\otimes 1/x^2, \ldots)$$

defines a non-zero element in the f-adic Tate module of $Q\otimes_R R'$.

Therefore, one whould have to prove more about the morphism

$$\Delta_{\mathcal{O}_C/p^n} \to \Delta_R.$$

[38] If $\alpha(\sum_{i=0}^{\infty} r_i s_i) = 0$, then for all $i \geq 0$ we get $(r_i, r_{i+1}) = (a_i x, a_i f)$ for some $a_i \in R$ because f, x is a regular sequence. From $a_i f = r_{i+1} = a_{i+1}x$ one derives

$$f^m a_0 \equiv f^{m-1}a_1 x \equiv \ldots \equiv a_m x^m \equiv 0 \bmod x^m$$

and therefore, using again that f,x is a regular sequence, that $a_0 \in (x^m)$ for all m. This forces $a_0 = 0$ and then $a_i = 0$ for all $i \geq 0$.

One may hope for example that Δ_R is a topologically free $\Delta_{\mathcal{O}_C/p^n}$-module, but this does not follow simply from (p,I)-complete faithful flatness : there exist (p,I)-completely faitfhfully flat $\Delta_{\mathcal{O}_C/p^n}$-modules which are not topologically free.

4.9. **Essential surjectivity.** Let R be quasi-regular semiperfectoid and let as before

$$\underline{M}_{\Delta}(-)\colon \mathrm{BT}(R)\to \mathrm{DF}_{\Delta}(R),\ G\mapsto (M_{\Delta}(G),\mathrm{Fil}M_{\Delta}(G),\varphi_{M_{\Delta}(G)})$$

be the prismatic Dieudonné functor with values in the category of filtered prismatic Dieudonné modules $\mathrm{DF}(R)$ (cf. Section 4.2 and Theorem 4.6.6).

Let us fix a perfect prism (A,I), a generator $\tilde{\xi}\in I$ and a surjection $A/I\twoheadrightarrow R$. Let $\xi:=\varphi^{-1}(\tilde{\xi})$. By Corollary 2.1.10 we may assume that A/I is henselian along $\ker(A/I\to R)$.

Let us first assume that $\ker(A/I\to R)$ is generated by some elements a_j, $j\in J$, that admit compatible systems $(a_j, j^{1/p}, a_j^{1/p^2},\ldots)$ of p^n-roots. Define

$$S:=A/I\langle X_j^{1/p^\infty}\mid j\in J\rangle/(X_j)$$

and $S\to R,\ X_j^{1/p^n}\mapsto \overline{a_j^{1/p^n}}$.

We note that by Theorem 3.3.9, we can always arrange this situation after passing to a quasi-syntomic cover of A/I[39].

Lemma 4.9.1. *The base change functor* $\mathrm{DF}(S)\to\mathrm{DF}(R)$ *on filtered prismatic Dieudonné modules is essentially surjective.*

Proof. By Proposition 4.1.25 it suffices to show that each pair consisting of a finite projective Δ_R-module M and an isomorphism $\varphi_M\colon \varphi^*M\cong M$ may be lifted to Δ_S. For this it suffices to see that $\Delta_S\to\Delta_R$ is surjective and henselian along its kernel (cf. Lemma 4.1.26). The surjectivity follows from the Hodge-Tate comparison as $L_{S/A/I}\to L_{R/A/I}$ is surjective by our assumption that the $a_j, j\in J$, generate $\ker(A/I\to R)$. First note that the pair $(S,\ker(S\to R))$ is henselian because the X_j^{1/p^n} are nilpotent in S and we assumed that A/I is henselian along $\ker(A/I\to R)$. To show that Δ_S is henselian along $K:=\ker(\Delta_S\to\Delta_R)$ it suffices to see $S\cong\Delta_S/\ker(\theta_S)$ is henselian along $\overline{K}:=(K+\ker(\theta))/\ker(\theta)$ (cf. [49, Tag 0DYD])). But $\overline{K}\subseteq S$ is contained in $\ker(S\to R)$. Another application of [49, Tag 0DYD] therefore implies that S is henselian along $\overline{K}$ because $(S,\ker(S\to R))$ is henselian. This finishes the proof. $\square$

Note that the ring

$$S=A/I\langle X_j^{1/p^\infty}\mid j\in J\rangle/(X_j\mid j\in J)$$

admits a surjection from the perfectoid ring[40]

$$\tilde{S}:=A/I[[X_j^{1/p^\infty}\mid j\in J]]\cong A[[X_j^{1/p^\infty}\mid j\in J]]/(\tilde{\xi})$$

by sending $X_j^{1/p^n}\mapsto X_j^{1/p^n}$.

[39]Note that by the (the proof of) [11, Lemma 4.24] any morphism from a perfectoid ring to a quasi-regular semiperfectoid ring is quasi-syntomic.

[40]More precisely, $\tilde{S}$ is the p-adic completion of $\varinjlim_n A/I[[X_j^{1/p^n}\mid j\in J]]$.

Lemma 4.9.2. *The natural functor*

$$\mathrm{DF}(\tilde{S}) \to \mathrm{DF}(S)$$

is essentially surjective.

Proof. The ring $\tilde{S}$ is henselian along $(X_j \mid j \in J)$. By using normal decompositions, i.e., Proposition 4.1.25, it suffices to see that the functor

$$\varphi - \mathrm{Mod}_{\Delta_{\tilde{S}}} \to \varphi - \mathrm{Mod}_{\Delta_S}$$

is essentially surjective. We note that by (a variant of) [12, Proposition 3.13] (and Proposition 3.4.2)

$$\Delta_S \cong \Delta_{\tilde{S}}\{\frac{X_j}{\tilde{\xi}} \mid j \in J\}^{\wedge}_{(p,\tilde{\xi})}$$

as the X_j form an infinite regular sequence in $\tilde{S}$. Define

$$B := \Delta_{\tilde{S}}/(X_j \mid j \in J),$$

where $\Delta_{\tilde{S}} \cong A[[X_j^{1/p^\infty} \mid j \in J]]$. Then B is p-torsion free and $\tilde{\xi}$-torsion free and thus defines a prism. Moreover, canonically $S \cong B/\tilde{\xi}$. By the universal property of Δ_S there exists therefore a canonical morphism

$$\alpha\colon \Delta_S \to B.$$

Concretely, the morphism α sends $X_j \mapsto 0$. Using a variant of Lemma 4.1.24 we see that Δ_S is henselian along $\ker(\alpha)$. By Lemma 4.9.3 $\varphi(\ker(\alpha)) \subseteq \tilde{\xi}\Delta_S$ and $\varphi/\tilde{\xi}$ is topologically nilpotent on $\ker(\alpha)$. Thus by Lemma 4.1.27 the categories of windows over Δ_S and B are equivalent. Therefore it suffices to see that windows over B can be lifted to windows over $\Delta_{\tilde{S}}$. After choosing a normal decomposition, this follows as the functor

$$\varphi - \mathrm{Mod}_{\Delta_{\tilde{S}}} \to \varphi - \mathrm{Mod}_B$$

is essentially surjective, which is true as Δ_S is henselian along the kernel of $\Delta_S \twoheadrightarrow B$ (cf. the end of the proof of Lemma 4.1.26). This finishes the proof. □

To finish the proof of Lemma 4.9.1 we have to prove the following lemmas.

Lemma 4.9.3. *With the notations from the proof of Lemma 4.9.2 we get* $\varphi(\ker(\alpha)) \subseteq \tilde{\xi}\Delta_S$ *and* $\varphi_1 := \varphi/\tilde{\xi}$ *is topologically nilpotent on* $\ker(\alpha)$.

Proof. Set $K := \ker(\alpha)$. Then K is the closure in the $(p, \tilde{\xi})$-adic topology of the Δ_S-submodule generated by $\delta^n(X_j/\tilde{\xi})$ for $j \in J$ and $n \geq 0$. By Lemma 4.9.4 the module K equals the closure of the ideal generated by

$$z_{j,n} := \frac{X_j^{p^n}}{\varphi^n(\tilde{\xi})\varphi^{n-1}(\tilde{\xi})^p \cdots \tilde{\xi}^{p^n}}$$

for $j \in J$ and $n \geq 0$. Let us show that $\varphi(K) \subseteq \tilde{\xi}\Delta_S$. Clearly,

$$\varphi(z_{j,n}) = \tilde{\xi}^{p^{n+1}} z_{j,n+1}. \tag{3}$$

As $\mathcal{N}^{\geq 1}\Delta_S$ is closed in Δ_S (being the kernel of the continuous surjection $\Delta_S \to S$), we can conclude $K \subseteq \mathcal{N}^{\geq 1}\Delta_S$. Next, let us check that φ_1 is topologically nilpotent on K. Fix $l \geq 1$. We claim that for every $m \geq 1$ such that $p^m > l$ and any $k \in K$ we have

$$\varphi_1^m(k) \in \tilde{\xi}^l K.$$

This implies as desired that φ_1 is topologically nilpotent on K. As $\tilde{\xi}^l K$ is closed and φ_1^m continuous (for the $(p, \tilde{\xi})$-adic topology on K) it is enough to assume that $k = z_{j,n}$ for some $j \in J, n \geq 1$, because the $z_{j,n}$ generate a dense submodule in K[41]. Using (Equation (3)) we can calculate

$$\varphi_1^m(z_{j,n}) = \varphi_1^{m-1}(\tilde{\xi}^{p^{n+1}-1} z_{j,n+1}) = \ldots = a\tilde{\xi}^{p^{n+m}-1} z_{j,n+m} \in \tilde{\xi}^{p^{n+m}-1} K$$

for some $a \in \mathbb{\Delta}_S$. But $\tilde{\xi}^{p^{n+m}-1} K \subseteq \tilde{\xi}^l K$ because $p^{n+m} - 1 \geq l$. This finishes the proof. □

Lemma 4.9.4. *Let (A, I) be a prism and let $d \in A$ be distinguished. Let furthermore $x \in A$ be an element of rank 1. Then for $n \geq 1$*

$$z_n := \frac{x^{p^n}}{\varphi^n(d)\varphi^{n-1}(d)^p \cdots d^{p^n}} \in A\{\frac{x}{d}\} := A\{z\}/(dz - x)_\delta$$

and the resulting morphism

$$A[y_1, y_2, \ldots]/(x - dy_1, y_1^p - \varphi(d)y_2, y_2^p - \varphi^2(d)y_3, \ldots) \to A\{\frac{x}{d}\}, \ y_n \mapsto z_n$$

is surjective.

By derived Nakayama the conclusion holds thus as well after (derived) (p, d)-adic completion.

Proof. We can argue in the universal case $A = \mathbb{Z}_p[x]\{d, \frac{1}{\delta(d)}\}^\wedge_{(p,d)}$ where $\delta(x) = 0$, thus we may assume that A is transversal, i.e., that (p, d) is a regular sequence in A, and that (x, d) is a regular sequence. This implies that for all $r \geq 1$ the sequence $(\varphi^r(d), \varphi^{r-1}(d))$ is regular as well (cf. Lemma 2.1.7). We first claim that for all $n \geq 0$ the element

$$z_n := \frac{x^{p^n}}{\varphi^n(d)\varphi^{n-1}(d)^p \cdots d^{p^n}}$$

lies in $A\{\frac{x}{d}\}$. If $n = 0$, then $z_n = \frac{x}{d} \in A\{\frac{x}{d}\}$. For $n \geq 0$ we can calculate

$$\varphi(z_n) = \frac{x^{p^{n+1}}}{\varphi^{n+1}(d) \cdots \varphi(d)^{p^n}}$$

because $\varphi(x) = x^p$. The numerator $x^{p^{n+1}}$ is divisible by $d^{p^{n+1}}$ in $A\{\frac{x}{d}\}$. As $(d^{p^{n+1}}, \varphi^{n+1}(d) \cdots \varphi(d)^{p^n})$ is a regular sequence in A we can conclude that $d^{p^{n+1}}$ divides $\frac{x^{p^{n+1}}}{\varphi^{n+1}(d)\cdots\varphi(d)^{p^n}}$, i.e., that $z_{n+1} \in A\{\frac{x}{d}\}$. Next we claim that the morphism

$$A[y_1, y_2, \ldots]/(x - dy_1, y_1^p - \varphi(d)y_2, y_2^p - \varphi^2(d)y_3, \ldots) \to A\{\frac{x}{d}\}, \ y_n \mapsto z_n$$

is surjective. For this it suffices to show for all $n \geq 0$ that $\delta^n(\frac{x}{d})$ lies in the subring $A[z_1, \ldots, z_{n+1}]$ of $A\{\frac{x}{d}\}$ generated by the $z_1, \ldots, z_{n+1}$. This claim follows from the assertion that $\delta(z_n) \in A[z_1, \ldots, z_{n+1}]$ using induction and how δ acts on sums and products. For $n = 0$ we can calculate

$$\delta(z_0) = \delta(\frac{x}{d}) = \frac{1}{p}(\varphi(\frac{x}{d}) - \frac{x^p}{d^p}) = \frac{1}{p}(d^p - \varphi(d))z_1 = \delta(d)z_1 \in A\{\frac{x}{d}\}.$$

[41] Dense for the $(p, \tilde{\xi})$-adic topology.

Similarly, we see

$$\delta(z_n) = \frac{1}{p}(d^{p^{n+1}} - \varphi^{n+1}(d))z_{n+1}$$

where the term $\frac{1}{p}(d^{p^{n+1}} - \varphi^{n+1}(d))$ lies in A. This finishes the proof. $\square$

We can derive essential surjectivity.

Theorem 4.9.5. *Let R be a quasi-regular semiperfectoid ring, which is flat over $\mathbb{Z}/p^n$ (for some $n > 0$) or over $\mathbb{Z}_p$. Then the filtered prismatic Dieudonné functor*

$$\underline{M}_{\Delta}(-) \colon \mathrm{BT}(R) \to \mathrm{DF}(R)$$

from the category of p-divisible groups over R to the category of filtered prismatic Dieudonné crystals over R is essentially surjective.

Proof. To prove the theorem, we may pass to a quasi-syntomic cover R' of R: indeed, let $\underline{M} \in \mathrm{DF}(R)$ such that its base change along the map $R \to R'$ is of the form $\underline{M}_{\Delta}(G')$, for some p-divisible group G' over R. The descent datum for $\underline{M}_{\Delta}(G')$ expressing that it comes from a filtered prismatic Dieudonné module over R (namely, $\underline{M}$) gives rise to a descent datum for G', since fully faithfulness over $R'\hat{\otimes}_R R'$ is already proved (cf. Theorem 4.8.5). This descent datum is effective, by p-completely faithfully flat descent for p-divisible groups (cf. Proposition A.11), so there exists a p-divisible group G over R, with $\underline{M}_{\Delta}(G) = \underline{M}$.

Therefore, by Theorem 3.3.9, we may and do assume that $R \cong A/I/(a_j \mid j \in J)$ for A/I a perfectoid ring and $a_j \in R$ admitting compatible systems of p^n-roots of unity. Using Lemma 4.9.1 we may even assume that

$$R \cong A/I\langle X_j^{1/p^\infty} \mid j \in J\rangle/(X_j).$$

In this case we can invoke Lemma 4.9.2 and reduce to the case that R is perfectoid. Then we can cite Corollary 4.3.8 to conclude that $\underline{M}_{\Delta}(-)$ is essentially surjective. $\square$

This concludes the proof of the main Theorem 4.6.9.

Remark 4.9.6. Let R be quasi-syntomic ring, flat over $\mathbb{Z}/p^n$ for some $n \geq 0$ or $\mathbb{Z}_p$. The filtered prismatic Dieudonné crystal of the étale p-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$ is given by

$$\underline{\mathcal{M}}_{\Delta}(\mathbb{Q}_p/\mathbb{Z}_p) = (\mathcal{O}^{\mathrm{pris}}, \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}, \varphi),$$

see Section 4.7. The functor $\mathcal{G}$ from $\mathrm{DF}(R)$ to the category of abelian sheaves of $(R)_{\mathrm{qsyn}}$, sending $\underline{\mathcal{M}} \in \mathrm{DF}(R)$ to

$$\mathcal{G}(\underline{\mathcal{M}}) = \mathcal{H}om_{\mathrm{DM}(R)}(\mathcal{M}, \mathcal{M}_{\Delta}(\mathbb{Q}_p/\mathbb{Z}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$$

(which only depends on the underlying prismatic Dieudonné crystal) defines a quasi-inverse of the filtered prismatic Dieudonné functor. Indeed, if $G \in \mathrm{BT}(R)$, one has a short exact sequence of abelian sheaves on $(R)_{\mathrm{qsyn}}$

$$0 \to T_p(G) \to \tilde{G} = T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to G \to 0$$

and for any $R' \in (R)_{\mathrm{qsyn}}$,

$$T_p(G)(R') = \mathrm{Hom}_{(R')_{\mathrm{qsyn}}}(\mathbb{Q}_p/\mathbb{Z}_p, G) = \mathrm{Hom}_{\mathrm{DM}(R')}(\mathcal{M}_{\Delta}(G), \mathcal{M}_{\Delta}(\mathbb{Q}_p/\mathbb{Z}_p)),$$

since, R' being itself quasi-syntomic over $\mathbb{Z}/p^n$ or $\mathbb{Z}_p$, the prismatic Dieudonné functor over R' is fully faithful. This shows that

$$G = \mathcal{G}(\underline{\mathcal{M}}_{\Delta}(G)).$$

By duality, one can rewrite the above formula more explicitely. Namely,

$$\mathcal{G}(\underline{\mathcal{M}}) = (\mathcal{M}^\vee)^{\varphi=1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p,$$

where $\mathcal{M}^\vee$ denotes the $\mathcal{O}^{\mathrm{pris}}$-linear dual of $\mathcal{M}$.

Nevertheless, it does not seem that these formulas are very useful in practice. It looks difficult to prove directly that $\mathcal{G}$ takes values in the category of (quasi-syntomic sheaves attached to) p-divisible groups. In the case of étale p-divisible groups Theorem 4.6.9 yields an equivalence of $\mathbb{Z}_p$-local systems on R and finite locally free $\mathcal{O}^{\mathrm{pris}}$-modules (resp. $\mathbb{\Delta}_R$-modules if R is quasi-regular semiperfectoid) $\mathcal{M}$ together with an isomorphism $\varphi_{\mathcal{M}} \colon \varphi^*(\mathcal{M}) \cong \mathcal{M}$. This is a generalization of Katz' correspondence between $\mathbb{Z}_p$-local systems on the spectrum $\mathrm{Spec}(k)$ of a perfect field k and φ-modules over $W(k)$ (cf. [27, Proposition 4.1.1.]. We thank Benoît Stroh for pointing this out to us.

5. Complements

5.1. **Prismatic Dieudonné theory for finite locally free group schemes.** Let R be a perfectoid ring. We fix a generator ξ of $\ker(\theta)$ and let $\tilde{\xi} = \varphi(\xi)$.

Definition 5.1.1. A *torsion prismatic Dieudonné module over* R is a triple

$$(M, \varphi_M, \psi_M),$$

where M is a finitely presented $A_{\mathrm{inf}}(R)$-module of projective dimension ≤ 1 which is annihilated by a power of p and where $\varphi_M : M \to M$ and $\psi_M : M \to M$ are respectively φ-linear and φ^{-1}-linear, and satisfy

$$\varphi_M \circ \psi_M = \tilde{\xi}, \quad \psi_M \circ \varphi_M = \xi.$$

The category of torsion prismatic Dieudonné modules over R is denoted by $\mathrm{DM}_{\mathrm{tors}}(R)$.

The base change of torsion prismatic Dieudonné modules behaves well.

Lemma 5.1.2. *Let $R \to R'$ be a morphism of perfectoid rings and $M \in \mathrm{DM}_{\mathrm{tors}}(R)$. Then $M \otimes_{A_{\mathrm{inf}}(R)} A_{\mathrm{inf}}(R')$ is concentrated in degree 0. In particular, the base change functor $\mathrm{DM}_{\mathrm{tors}}(R) \to \mathrm{DM}_{\mathrm{tors}}(R')$ is exact.*

Proof. Let

$$0 \to M_1 \xrightarrow{f} M_2 \to M \to 0$$

be a resolution of M by finite locally free $A_{\mathrm{inf}}(R)$-modules. As M is killed by p^n for some $n \geq 0$, there exists $g \colon M_2 \to M_1$ such that $f \circ g = p^n$. Then $p^n = g \circ f$ (using that f is injective). The base change $M_1 \otimes_{A_{\mathrm{inf}}(R)} A_{\mathrm{inf}}(R')$ is p-torsion free as $A_{\mathrm{inf}}(R')$ is. This implies that the base change of f to $A_{\mathrm{inf}}(R')$ remains injective, which finishes the proof. □

Before stating the main result, let us introduce a notation, which will be in use only in this section.

Notation 5.1.3. If S is a p-complete ring, let $\mathcal{B}_S$ (resp. $\mathcal{C}_S$) denote the category whose objects are $\mathcal{O}_{\Delta}$-modules on $(S)_{\Delta}$ (resp. $\mathcal{O}_{\Delta}$-modules on $(S)_{\Delta}$ endowed with a φ-linear Frobenius), and whose morphisms are $\mathcal{O}_{\Delta}$-linear morphisms (resp. $\mathcal{O}_{\Delta}$-linear morphisms commuting with Frobenius).

Theorem 5.1.4. *There is a natural exact*[42] *antiequivalence*

$$H \mapsto (M_{\Delta}(H), \varphi_{M_{\Delta}(H)}, \psi_{M_{\Delta}(H)})$$

between the category of finite locally free group schemes of p-power order on R and the category $\mathrm{DM}_{\mathrm{tors}}(R)$ of torsion prismatic Dieudonné modules over R, such that the $A_{\mathrm{inf}}(R)$-module $M_{\Delta}(H)$ is given by the formula

$$M_{\Delta}(H) = \mathrm{Ext}^1_{(R)_{\Delta}}(u^{-1}H, \mathcal{O}_{\Delta})$$

and such that $\varphi_{M_{\Delta}(H)}$ is the map induced by the Frobenius of $\mathcal{O}_{\Delta}$.

Remark 5.1.5. A similar statement can be found in [36, Theorem 10.12]. Apart from the change of terminology, the only difference with the result in loc. cit. is that we remove the assumption that $p \geq 3$ and provide a formula for the underlying A_{inf}-module of the torsion minuscule Breuil-Kisin-Fargues module attached to a finite locally free group scheme of p-power order.

[42] This includes the non-formal assertion that the inverse equivalence is exact, too.

The proof of Theorem 5.1.4 will make use of the following lemma.

Lemma 5.1.6. *Let (A, I) be a bounded prism, such that A is p-torsion free and let S be a p-completely syntomic A/I-algebra*[43]*. Then*

$$H^0(S, \Delta_{S/A})$$

is p-torsion free.

Proof. As S is a p-completely syntomic A/I-algebra the derived prismatic cohomology $\Delta_{S/A}$ agrees with the cohomology $R\Gamma((S/A)_{\Delta}, \mathcal{O}_{\Delta})$ of the prismatic site of S over A (this follows by descent from the quasi-regular semiperfectoid case and Proposition 3.4.2). By [12, Proposition 3.13] and the assumption that S is a p-completely syntomic A/I-algebra, one can calculate $\Delta_{S/A}$ by some Čech-Alexander complex that has p-complete p-completely flat terms over A. Therefore it suffices to see that each p-complete p-completely flat A-algebra B has no p-torsion. As A is p-torsion free, A, and thus B, is p-completely flat over $\mathbb{Z}_p$. But any p-completely flat p-complete module over $\mathbb{Z}_p$ is topologically free and thus p-torsion free. □

Proof of Theorem 5.1.4. The construction of the antiequivalence is exactly similar to the one of [36, Theorem 10.12], replacing Theorem 9.8 in loc. cit. by Corollary 4.3.8, so we do not give it and refer the reader to [36]. The simple principle is that Zariski-locally on $\mathrm{Spec}(R)$, any finite locally free group scheme of p-power order is the kernel of an isogeny of p-divisible groups (and even an isogeny of p-divisible groups associated to abelian schemes, cf. Theorem 4.6.1) ; similarly, Zariski-locally on $\mathrm{Spec}(R)$, any torsion prismatic Dieudonné module is the cokernel of an isogeny of prismatic Dieudonné modules ([36, Lemma 10.10]).

Let us now prove that

$$M_{\Delta}(H) = \mathrm{Ext}^1_{(R)_{\Delta}}(u^{-1}H, \mathcal{O}_{\Delta})$$

and that the functor $M_{\Delta}(-)$ preserves exactness for a short exact sequence

$$0 \to H' \to H \to H'' \to 0$$

of finite locally free group schemes of p-power order over R. Note that this implies by Mittag-Leffler exactness of

$$0 \to M_{\Delta}(H') \to M_{\Delta}(H) \to M_{\Delta}(H'') \to 0$$

if H', H, H'' are finite locally free group schemes of p-power order or p-divisible groups.

By construction of the antiequivalence, it suffices to check that if H is the kernel of an isogeny $X \to X'$, with X, X' are abelian schemes over R, the natural map

$$M_{\Delta}(X[p^\infty]) = \mathrm{Ext}^1_{(R)_{\Delta}}(u^{-1}X, \mathcal{O}_{\Delta}) \to \mathrm{Ext}^1_{(R)_{\Delta}}(u^{-1}H, \mathcal{O}_{\Delta})$$

is surjective. But the cokernel of this map embeds in $\mathrm{Ext}^2_{(R)_{\Delta}}(u^{-1}X', \mathcal{O}_{\Delta})$, which is zero by Theorem 4.5.6.

For exactness, start with a short exact sequence of finite locally free group schemes of p-power order on R

$$0 \to H' \to H \to H'' \to 0,$$

[43] A morphism $R \to R'$ between p-complete rings of bounded p^∞-torsion is p-completely syntomic if $R'/p \cong R' \otimes^{\mathbb{L}}_R R/p$ and $R/p \to R'/p$ is syntomic in the sense of [49, Tag 00SL].

which we see as an exact sequence of abelian sheaves on $(R)_{\mathrm{qsyn}}$. The surjectivity of the map

$$M_{\Delta}(H) \to M_{\Delta}(H')$$

can be checked locally and so we can assume that H, and so also H', embeds in an abelian scheme X. But we know that the map

$$M_{\Delta}(X[p^\infty]) \to M_{\Delta}(H')$$

is already surjective, again because $\mathrm{Ext}^2_{(R)_{\Delta}}(u^{-1}X/H', \mathcal{O}_{\Delta}) = 0$. Thus, the same holds for the map

$$M_{\Delta}(H) \to M_{\Delta}(H').$$

To prove injectivity of the map

$$M_{\Delta}(H'') \to M_{\Delta}(H),$$

it suffices by the long exact sequence for $R\mathrm{Hom}_{(R)_{\Delta}}(-, \mathcal{O}_{\Delta})$ to prove that

$$\mathrm{Hom}_{(R)_{\Delta}}(u^{-1}H', \mathcal{O}_{\Delta}) = 0.$$

Let us prove that $\mathrm{Hom}_{(R)_{\Delta}}(u^{-1}H', \mathcal{O}_{\Delta})$ is p-torsion free. This is enough : indeed, we know it is also killed by a power of p, because $u^{-1}H'$ is. As

$$\mathrm{Hom}_{(R)_{\Delta}}(u^{-1}H', \mathcal{O}_{\Delta}) \subset H^0(u^{-1}H', \mathcal{O}_{\Delta}) = H^0(H', \Delta_{H'/A_{\mathrm{inf}}}),$$

it suffices to prove that the latter is p-torsion free. This is the content of Lemma 5.1.6 when applied to the p-completely syntomic R-scheme H'.

Let

$$\mathcal{G}\colon \mathrm{DM}_{\mathrm{tors}} \to \{\text{finite locally free group schemes of } p\text{-power order over } R\}$$

be an inverse functor to $M_{\Delta}(-)$. We claim that $\mathcal{G}$ is exact. Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence in $\mathrm{DM}_{\mathrm{tors}}(R)$. For any morphism $R \to R'$ the base change of it along $A_{\mathrm{inf}}(R) \to A_{\mathrm{inf}}(R')$ will stay exact by 5.1.2. By [19, Proposition 1.1.] and compatibility of $\mathcal{G}$ with base change in R we can therefore assume that R is a perfect field of characteristic p. In this case the category of finite locally free group schemes of p-power order and the category $\mathrm{DM}_{\mathrm{tors}}$ are abelian and thus any equivalence between them is automatically exact. □

Remark 5.1.7. Let H be a finite locally free group scheme of p-power order over the perfectoid ring R. We have seen in the previous proof that

$$\mathrm{Hom}_{(R)_{\Delta}}(u^{-1}H, \mathcal{O}_{\Delta}) = 0.$$

Assume $pR = 0$, i.e. that R is a perfect $\mathbb{F}_p$-algebra. This result is to be contrasted with the fact that

$$\mathrm{Hom}_{(R/\mathbb{Z}_p)_{\mathrm{crys}}}((u^{\mathrm{crys}})^{-1}H, \mathcal{O}_{\mathrm{crys}})$$

is far from being 0. It is isomorphic to the $\mathcal{O}_{\mathrm{crys}}$-linear dual of the crystalline Dieudonné module

$$M_{\mathrm{crys}}(\check{H}) = \mathcal{E}xt^1_{(R/\mathbb{Z}_p)_{\mathrm{crys}}}((u^{\mathrm{crys}})^{-1}H, \mathcal{O}_{\mathrm{crys}})$$

of the Cartier dual of H. In fact, one has a natural isomorphism ([5, Theorem 5.2.7])

$$\tau_{\leq 1} R\mathcal{H}om_{(R/\mathbb{Z}_p)_{\mathrm{crys}}}((u^{\mathrm{crys}})^{-1}H, \mathcal{O}_{\mathrm{crys}})^\vee[-1] \simeq \tau_{\leq 1} R\mathcal{H}om_{(R/\mathbb{Z}_p)_{\mathrm{crys}}}((u^{\mathrm{crys}})^{-1}\check{H}, \mathcal{O}_{\mathrm{crys}}),$$

where

$$(-)^\vee := R\mathcal{H}om_{\mathcal{O}_{\mathrm{crys}}}(-, \mathcal{O}_{\mathrm{crys}}).$$

The previous equality is deduced from this by taking cohomology in degree 0. Similarly, one has a natural isomorphism

$$\tau_{\leq 1} R\mathcal{H}om_{(R)_{\Delta}}(u^{-1}H, \mathcal{O}_{\Delta})^\vee[-1] \simeq \tau_{\leq 1} R\mathcal{H}om_{(R)_{\Delta}}(u^{-1}\check{H}, \mathcal{O}_{\Delta}),$$

where

$$(-)^\vee := R\mathcal{H}om_{\mathcal{O}_{\Delta}}(-, \mathcal{O}_{\Delta}).$$

One does not see anything interesting by taking cohomology in degree 0 on both sides. But in degree 1, one obtains

$$\mathcal{E}xt^1_{\mathcal{O}_{\Delta}}(\mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}H, \mathcal{O}_{\Delta}), \mathcal{O}_{\Delta}) \simeq \mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}\check{H}, \mathcal{O}_{\Delta}),$$

and therefore

$$\mathrm{Ext}^1_{\Delta_R}(M_{\Delta}(H), \Delta_R) \simeq M_{\Delta}(\check{H})$$

expressing the compatibility of the functor M_{Δ} with Cartier duality on the category of finite locally free group schemes of p-power order.

Remark 5.1.8. Let R be quasi-syntomic ring, flat over $\mathbb{Z}/p^n$ or $\mathbb{Z}_p$. Although the same trick allows in principle to deduce from Theorem 4.6.9 a classification result for finite locally free group schemes of p-power order over R, it seems more subtle to obtain a nice description of the target category, i.e. of the objects which can locally on R be written as the cokernel of an isogeny of filtered prismatic Dieudonné crystals on R. At least the arguments given above should go through whenever the forgetful functor

$$\mathrm{DF}(R) \to \mathrm{DM}(R)$$

is an equivalence, like in the case of perfectoid rings or in the Breuil-Kisin case to be discussed in the next section (where the classification of finite flat group schemes is already known, and was proved by Kisin following the same technique, cf. [30, Section 2.3.]).

5.2. **Comparison over $\mathcal{O}_K$.** In this section, we want to extract from Theorem 4.6.9 a concrete classification of p-divisible groups over p-complete regular local rings with perfect residue field of characteristic p. This will in particular recover Breuil-Kisin's classification ([14], [30]), as extended to all p by Kim [29], Lau [35] and Liu [38], over $\mathcal{O}_K$, for a complete discretely valued extension of $\mathbb{Q}_p$ with perfect residue field.

Proposition 5.2.1. *Let R be a p-complete Noetherian ring. If R is regular, there exists a quasi-syntomic perfectoid cover R_∞ of R.*

Proof. The existence of a faithfully flat cover $R \to R_\infty$, with R_∞ perfectoid, is explained in [9, Ex. 3.8]. Assume first that $pR = 0$ or that R is unramified. Since R is in particular an integral domain, R is either flat over $\mathbb{Z}_p$ or $pR = 0$. In the first case set $\Lambda := \mathbb{Z}_p$ and in the second $\Lambda := \mathbb{F}_p$. By [49, Tag 07GB] the morphism $\Lambda \to R$ is a filtered colimit of smooth ring maps and thus $L_{R/\Lambda}$ has p-complete Tor-amplitude in degree 0. The triangle attached to the composite $\Lambda \to R \to R_\infty$ shows that $L_{R_\infty/R}$ has p-complete Tor-amplitude in degree -1. Therefore the map $R \to R_\infty$ is indeed a quasi-syntomic cover. Finally, when R is ramified of mixed characteristic, one sees from the explicit construction of [9, Ex. 3.8 (5)] that $R \to R_\infty$ is the p-completion of a colimit of syntomic morphisms (obtained by extracting pth-roots), hence is quasi-syntomic. □

Remark 5.2.2. Conversely, the main result of [9] asserts that a Noetherian ring with p in its Jacobson radical which admits a faithfully flat map to a perfectoid ring has to be regular (this is a generalization of a theorem of Kunz [31] in positive characteristic).

Proposition 5.2.3. *Let R be a p-complete regular ring. The forgetful functor*

$$\mathrm{DF}(R) \to \mathrm{DM}(R)$$

from filtered prismatic Dieudonné crystals to prismatic Dieudonné crystals is an equivalence.

Proof. The forgetful functor is obviously faithful. Since R is quasi-syntomic and either p-torsion free or such that $pR = 0$, we know by Theorem 4.8.5 (and quasi-syntomic descent) that the composite functor

$$\mathrm{BT}(R) \to \mathrm{DF}(R) \to \mathrm{DM}(R)$$

is fully faithful. This implies that the forgetful functor over R is also full.

Let us prove essential surjectivity. Let $(\mathcal{M}, \varphi_{\mathcal{M}}) \in \mathrm{DM}(R)$. Let R_∞ be a perfectoid quasi-syntomic cover of R, as in Proposition 5.2.1. Let $\mathcal{M}_\infty \in \mathrm{DM}(R_\infty)$ be the base change of $\mathcal{M}$, which we see as a prismatic Dieudonné module M_∞ over R_∞, via the equivalence of Proposition 4.1.13. We know (Lemma 4.1.15) that the forgetful functor

$$\mathrm{DF}(R_\infty) \to \mathrm{DM}(R_\infty)$$

is an equivalence. In particular, $M_\infty \in \mathrm{DM}(R_\infty)$ acquires a unique filtration $\mathrm{Fil}M_\infty$ such that

$$\underline{M}_\infty = (M_\infty, \mathrm{Fil}M_\infty, \varphi_{M_\infty})$$

is a filtered prismatic Dieudonné module over R_∞. We want to see that $\mathrm{Fil}M_\infty$ descends, i.e., that the images of $\mathrm{Fil}M_\infty$ along the two base change maps

$$\mathbb{\Delta}_{R_\infty} \to \mathbb{\Delta}_{R'_\infty},$$

where $R'_\infty = R_\infty \hat{\otimes}_R R_\infty$, coincide. This holds because a prismatic Dieudonné module over R'_∞ admits at most one filtration making it a filtered prismatic Dieudonné module, i.e., because the forgetful functor

$$\mathrm{DF}(R'_\infty) \to \mathrm{DM}(R'_\infty)$$

is fully faithful. Indeed, R'_∞ is quasi-regular semiperfectoid and either p-torsion free or of characteristic p (since it is p-completely faithfully flat over the regular ring R) ; therefore, exactly as above for R, we know that the forgetful functor over R'_∞ is fully faithful (using again Theorem 4.8.5). □

Recall the following definition, which already appeared in Example 4.1.21 before.

Definition 5.2.4. Let $(A, I = (d))$ be a prism. A *Breuil-Kisin module* (M, φ_M) *over* (A, I)*, or just* A *if* I *is understood*, is a finite free A-module M together with an isomorphism

$$\varphi_M : \varphi^* M[\frac{1}{I}] \cong M[\frac{1}{I}].$$

If $\varphi_M(\varphi^* M) \subseteq M$ with cokernel killed by I and finite projective as an A/I-module, then (M, φ_M) is called *minuscule*.

We denote by $\mathrm{BK}(A)$ the category of Breuil-Kisin modules over A and by $\mathrm{BK}_{\min}(A) \subseteq \mathrm{BK}(A)$ its full subcategory of minuscule ones.

We now specialize the previous discussion to the case where R is a complete regular local ring with perfect residue field k of characteristic p. Any such R can be written as

$$R = W(k)[[u_1, \dots, u_d]]/(E),$$

where $d = \dim R$ and E is a power series with constant term of p-value one (cf. [40, Theorem 29.7, Theorem 29.8 (ii)]). Let (A, I) be the prism

$$(A, I) = (W(k)[[u_1, \dots, u_d]], (E)),$$

where the δ-ring structure on A is the usual one on $W(k)$ and is such that $\delta(u_i) = 0$, for $i = 1, \dots, d$. For simplicity, we assume $d = 1$ in the following, but the general case works similarly.

Remark 5.2.5. Since R and A are regular, the condition that the cokernel is projective over $A/(E) = R$ in the definition of a minuscule Breuil-Kisin module over A is automatic, as can be seen using the Auslander-Buchsbaum formula.

Theorem 5.2.6. *Let R be a complete regular local ring with perfect residue field of characteristic p. The functor*

$$\mathrm{BT}(R) \to \mathrm{BK}_{\min}(A) \quad ; \quad G \mapsto v^*\mathcal{M}_{\Delta}(G)((A, I)) = \mathcal{E}xt^1_{(R)_{\Delta}}(u^{-1}G, \mathcal{O}_{\Delta})_{(A,I)}$$

is an equivalence of categories.

The case where $pR = 0$ follows from Corollary 4.3.3, the classical fact that a Dieudonné crystal over R is the same thing as a minuscule Breuil-Kisin module over A (with respect to p) together with an integrable topologically quasi-nilpotent connection making Frobenius horizontal and [16, Proposition 2.7.3], which proves that for this particular ring A, the connection is necessarily unique. Hence in the following, we will always assume that R is p-torsion free. In this case, the pair (p, E) is transversal.

Remark 5.2.7. When $R = \mathcal{O}_K$, with K a complete discretely valued extension of $\mathbb{Q}_p$ with perfect residue field, A is usually denoted by $\mathfrak{S}$ (a notation which seems to originate from [14]). We will see below that the antiequivalence of the theorem coincides in this case with the one studied by Kisin for p odd and Kim, Lau and Liu when $p = 2$.

We will describe prismatic Dieudonné crystals over $\mathcal{O}_K$ via descent using the following lemma.

Lemma 5.2.8. *The natural map from the sheaf represented by (A, I) to the final object of $\mathrm{Shv}((R)_{\Delta})$ is an epimorphism for the p-completely faithfully flat topology.*

Proof. Indeed, let $(B, J) \in (R)_{\Delta}$. Let A_∞ be the perfection of A ; the map $R = A/I \to R_\infty = A_\infty/IA_\infty$ is a quasi-syntomic cover. By base change, the map

$$B/J \to B/J\hat{\otimes}_R R_\infty$$

is therefore a quasi-syntomic cover as well. By Proposition 3.3.8 there exists a prism (C, JC) which is p-completely faithfully flat over (B, J) such that there exists a morphism of B/J-algebras $B/J\hat{\otimes}_R R_\infty \to C/J$. Since R_∞ is perfectoid, it implies that (C, JC) lives over (A_∞, IA_∞) (cf. Proposition 2.1.11), and a fortiori over (A, I), as desired. $\square$

Proof of Theorem 5.2.6. By Theorem 4.6.9 and Proposition 5.2.3, we know that the prismatic Dieudonné functor

$$\mathcal{M}_{\mathbb{\Delta}} : \mathrm{BT}(R) \to \mathrm{DM}(R)$$

is an antiequivalence. Therefore, it suffices to prove that the functor

$$\mathcal{M} \to v^*\mathcal{M}((A, I))$$

from prismatic Dieudonné crystals $\mathrm{DM}(R)$ to minuscule Breuil-Kisin modules $\mathrm{BK}_{\min}(A)$ is an equivalence. Let B the absolute product of A with itself in $(R)_{\mathbb{\Delta}}$. One has (cf. [12, Proposition 3.13])

$$B = \big(W(k)[[u]] \otimes_{W(k)} W(k)[[v]]\big)\{\frac{u-v}{E(u)}\}_{\delta}^{\wedge(p,E(u))}$$

where we wrote $E(u)$ for $E \otimes 1$[44]. By Lemma 5.2.8 below and Proposition 4.1.8, a prismatic Dieudonné crystal $\mathcal{M}$ over R is the same thing as a minuscule Breuil-Kisin module N over A, together with a descent datum, i.e., an isomorphism

$$N \otimes_{A,p_1} B \cong N \otimes_{A,p_2} B$$

(where $p_1, p_2 : A \to B$ are the two natural maps), satisfying the usual cocycle condition.

We claim that any $N \in \mathrm{BK}_{\min}(A)$ comes with a unique descent datum. Indeed, let $f\colon B \to A$ be the map extending the multiplication map $A \otimes_{W(k)} A \to A$. Proposition 5.2.10 below shows that base change along f induces an equivalence between $\mathrm{BK}_{\min}(B)$ and $\mathrm{BK}_{\min}(A)$. In particular, this base change functor is fully faithful, and so it suffices to produce the descent datum after base change along f. But $f \circ p_1 = f \circ p_2 = \mathrm{Id}_A$, so one can simply take the isomorphism corresponding to the identity of N. The same argument shows that the descent datum is unique. $\square$

The proof of Proposition 5.2.10 relies on the following technical lemma.

Lemma 5.2.9. *With the notation from the proof of Theorem 5.2.6 the ideal $J \subseteq B$ is contained in $\mathcal{N}^{\geq 1}B$, stable by φ_1 and φ_1 is topologically nilpotent on J, with respect to the (p, E)-adic topology.*

Proof. Write $E := E(u)$. The ideal J is generated (up to completion) by the δ-translates of

$$z := (u - v)/E,$$

so to check that $J \subset \mathcal{N}^{\geq 1}B$, it is enough to prove that $\delta^n(z) \in \mathcal{N}^{\geq 1}B$ for all n. We prove by induction on n that for all $k \geq 0$, $\varphi^k(\delta^n(z))$ is divisible by E. For $n = 0$, one has, for any $k \geq 1$,

$$\varphi^k(z) = \frac{u^{pk} - v^{pk}}{\varphi^k(E)} = \frac{(u-v)(u^{pk-1} + u^{pk-2}v + \cdots + uv^{pk-2} + v^{pk})}{\varphi^k(E)}.$$

Since $(E, \varphi^k(E))$ is regular (as (p, E) is transversal because B is p-completely faithfully flat over $W(k)[[u]]$ by [12, Proposition 3.13]) and $u - v$ is divisible by E in B, we deduce that E divides $\varphi^k(z)$. Let now $n \geq 0$ and assume the result is known for $\delta^n(z)$. We have, for $k \geq 0$,

$$p\varphi^k(\delta^{n+1}(z)) = \varphi^k(p\delta^{n+1}(z)) = \varphi^k(\varphi(\delta^n(z)) - \delta^n(z)^p) = \varphi^{k+1}(\delta^n(z)) - \varphi^k(\delta^n(z))^p,$$

[44] If similarly, $E(v) = 1 \otimes E$, then $E(u)/E(v)$ is a unit in B by [12, Lemma 2.24] because $E(u)$ divides $E(v)$ in B. Namely, $E(v) = E(u)(\frac{E(v)-E(u)}{E(u)} + 1)$ in B and $u - v$ divides $E(u) - E(v)$.

so the statement for $\delta^{n+1}(z)$ follows by induction hypothesis, and the fact that p and E are transversal. This concludes the proof that $J \subset \mathcal{N}^{\geq 1}B$.

Let $x \in J$. We have

$$E.f(\varphi_1(x)) = f(\varphi(x)) = \varphi(f(x)) = 0.$$

Since E is a non-zero divisor in A, we must have $f(\varphi_1(x)) = 0$ and therefore $\varphi_1(x) \in J$, i.e., φ_1 stabilizes J.

It remains to prove that the divided Frobenius is topologically nilpotent on J, endowed with the (p, E)-adic topology. Let

$$A' = A\left\{\frac{\varphi(E)}{p}\right\}^{\wedge_p},$$

which by [12, Lemma 2.35] identifies with the (p-completed) divided power envelope $D_A((E))^{\wedge_p}$ of A in (E). The composition

$$\alpha : A \xrightarrow{\varphi} A \to A'$$

defines a morphism of prisms $(A, (E)) \to (A', (p))$. Let

$$B' := D_{A\hat{\otimes}_{W(k)}A}(J')^{\wedge_p},$$

where J' is the kernel of the map $A\hat{\otimes}_{W(k)}A \to R$. The ideal J' is generated by E and $u - v$, which form a regular sequence in $A\hat{\otimes}_{W(k)}A/p$, and therefore

$$\begin{aligned} B' \cong (A\hat{\otimes}_{W(k)}A)\left\{\frac{\varphi(E), \varphi(u-v)}{p}\right\}_{\delta}^{\wedge_p} &\cong (A\hat{\otimes}_{W(k)}A)\left\{\frac{p, \varphi(u-v)}{\varphi(E)}\right\}_{\delta}^{\wedge_{\varphi(E)}} \\ &\cong D_{\varphi^*_{A\hat{\otimes}_{W(k)}A}B}((E))^{\wedge_p}. \end{aligned}$$

(In the second isomorphism we used again [12, Lemma 2.24], and in the first and last [12, Lemma 2.37].) In particular, the map α induces a map, which we still denote by the same letter :

$$\alpha : B \to B'.$$

It sends $J \subseteq B$ to the kernel $K \subset B'$ of the map $B' \to A'$ (which extends the multiplication on $A\hat{\otimes}_{W(k)}A \to A$), and commutes with the divided Frobenius (because B' is p- and thus $\varphi(E)$-torsion free). The ideal $K \subseteq B'$ is generated (up to completion) by $(u - v)$ and the δ-translates of

$$\frac{\varphi(u-v)}{p} = \text{unit} \cdot \frac{\varphi(u-v)}{\varphi(E)}.$$

As the kernel J of $B \to A$ is stable by φ_1, this implies that $K = JB'$ is stable by φ_1, and thus in particular contained in $\mathcal{N}^{\geq 1}B'$.

Observe also that

$$pB' \cap B = (p, E).B$$

To see this, one needs to show that the map induced by α

$$B/(p, E) \to B'/p$$

is injective, i.e., by faithful flatness of $\varphi\colon A \to A$ that the natural map

$$B/(p, \varphi(E)) = B/(p, E^p) \to B'/p = D_B((E))/p$$

is injective. But since B is p-torsion free,

$$B'/p = B/(p, E^p)[X_0, X_1, \ldots]/(X_0^p, X_1^p, \ldots)^{\wedge p}$$

and the above map is simply the natural inclusion map. Hence, it suffices to prove topological nilpotence of $\varphi_1 =$ "$\varphi/\varphi(E)$" on K with respect to the p-adic topology[45]. We do it in two steps.

Note first that φ is topologically nilpotent on K. More precisely, using that K is stable by φ_1, one easily sees by induction that $\varphi^k(z)$ is divisible by p^k, for all $z \in K$ and $k \geq 1$ (with $\varphi^k(z)/p^k \in K$, because A' is p-torsion free). The equality

$$\varphi_1(xy) = \varphi(x)\varphi_1(y)$$

for $x, y \in K$, implies by induction that for any $n \geq 1$:

$$\varphi_1^n(xy) = \varphi^n(x)\varphi_1^n(y).$$

This shows that the second divided power ideal $K^{[2]}$ is stable by φ_1 (since K is stable by φ,φ_1) and, by what we just said, that the left hand side is divisible by p^n in K. In fact, one can do better. Let $m \geq 1$ and $x \in K$. In the previous equality, take $y = x^{m-1}$. Seeing it in $B'[1/p]$ (recall that B' is p-torsion free), one can divide both sides by $m!$. It reads :

$$\varphi_1^n(\gamma_m(x)) = \frac{\varphi^n(x)}{m!}\varphi_1^n(x^{m-1}).$$

The left hand side always makes sense in K since K has divided powers, and for n big enough, the right hand side as well since $\varphi^n(x)$ tends p-adically to 0 and thus is divisible by $m!$ for n big enough. Letting n go to infinity, we see that the left hand side goes to 0 in K. These considerations prove that φ_1 is topologically nilpotent (with respect to the p-adic topology) on $K^{[2]}$, as it is topologically nilpotent on K^2 and all divided powers $\gamma_m(x)$, $m \geq 2$, for $x \in K$.

Let e be the degree of the polynomial E. Since $K^{[2]}$ is stable by φ_1, φ_1 defines a semi-linear endomorphism of the quotient $K/K^{[2]}$. Let us now prove that $\varphi_1^{pe}(K/K^{[2]}) \subset p.K/K^{[2]}$. We know that the A'-module $K/K^{[2]}$ is isomorphic to $(\Omega^1_A)^{\wedge_p} \otimes_A A'$ (where the map $A \to A'$ is the natural inclusion). It is a free A'-module of rank generated by du and via this identification, one has $\varphi_1(du) = u^{p-1}du$. But the image of u^{pe} in A' is divisible by p since p divides E^p in A' and E is an Eisenstein polynomial. Therefore p (even p^{p-1}) divides $\varphi_1^{pe}(du \otimes 1)$ in $K/K^{[2]}$.

Finally, let us check that these two steps imply the desired topological nilpotence. Let $x \in K$, $\bar{x}$ its class in $K/K^{[2]}$. Fix an integer $n \geq 1$. By the second step, we have

$$\varphi_1^{pne}(\bar{x}) \in p^n K/K^{[2]},$$

i.e., there exists $y \in K^{[2]}$ such that

$$\varphi_1^{pne}(x) \in y + p^n K.$$

By the first step, there exists $m \geq 1$ such that $\varphi_1^m(y) \in p^n K$, and so

$$\varphi_1^{pne+m}(x) \in p^n K,$$

as desired. $\square$

[45]Let us clarify what we mean by the various φ_1's, whenever they are defined. On A we set $\varphi_1 = \varphi/E$ which is the restriction of $\varphi_1 = \varphi/\varphi(E)$ along α. In B' the element $\varphi(E)/p$ is a unit and thus $\varphi_1 = \frac{p}{\varphi(E)}\frac{\varphi}{p}$, i.e., both possible definition of the divided Frobenius differ by a unit.

Proposition 5.2.10. *With the notations from the proof of Theorem 5.2.6 the map $f\colon B \to A$ induces an equivalence:*

$$\mathrm{BK}_{\min}(B) \to \mathrm{BK}_{\min}(A).$$

Proof. From Lemma 5.2.9 and the fact that the map $B \to A$ is surjective, one deduces fully faithfulness of the functor

$$\mathrm{BK}_{\min}(B) \to \mathrm{BK}_{\min}(A)$$

as in the proof of Lemma 4.1.27. For essential surjectivity, observe that the map $\iota\colon A \to B$ sending a to the image of $a \otimes 1$ in B is a section of the map $B \to A$. Therefore, any $M \in \mathrm{BK}_{\min}(A)$ is the image by the functor

$$\mathrm{BK}_{\min}(B) \to \mathrm{BK}_{\min}(A)$$

of $\iota^* M \in \mathrm{BK}_{\min}(A)$. □

Finally, let K be a complete, discretely valued extension of $\mathbb{Q}_p$, let $\mathcal{O}_K \subseteq K$ be its ring of integers and assume the residue field k of $\mathcal{O}_K$ is perfect. We will show that the equivalence of Theorem 5.2.6 coincides with the equivalence established by Kisin (cf. [30, Theorem 0.4]). Set

$$\mathfrak{S} := W(k)[[u]]$$

with Frobenius lift $\varphi\colon W(k)[[u]] \to W(k)[[u]]$ sending $u \mapsto u^p$. Fix a uniformizer $\pi \in \mathcal{O}_K$ and define the morphism

$$\tilde{\theta}\colon \mathfrak{S} \to \mathcal{O}_K,\ u \mapsto \pi.$$

Then the kernel $\ker(\tilde{\theta}) = (E)$ is generated by an Eisenstein polynomial $E \in W(k)[u]$. Let S be the p-completed divided power envelope of the ideal $(E) \subseteq \mathfrak{S}$, i.e.,

$$S = \mathfrak{S}\{\frac{\varphi(E)}{p}\}^{\wedge}_{p}$$

in the category of δ-rings. Note that the composition

$$\psi_K\colon \mathfrak{S} \xrightarrow{\varphi} \mathfrak{S} \to S$$

induces to a morphism $(\mathfrak{S},(E)) \to (S,(p))$ of prisms. Via the composition $\mathcal{O}_K \cong \mathfrak{S}/(E) \xrightarrow{\psi_K} S/(p)$ we consider $(S,(p))$ as an object of the (absolute) prismatic site $(\mathcal{O}_K)_{\Delta}$. The antiequivalence

$$M^{\mathrm{Kis}}(-)\colon \mathrm{BT}(\mathcal{O}_K) \cong \mathrm{BK}_{\min}(\mathcal{O}_K)$$

of Kisin has the characteristic property (cf. [30, Theorem (2.2.7)]) that for a p-divisible group G over $\mathcal{O}_K$ there is a canonical Frobenius equivariant isomorphism

$$M^{\mathrm{Kis}}(G) \otimes_{\mathfrak{S},\psi} S \cong \mathbb{D}(G)(S)$$

where the right hand side denotes the evaluation of the crystalline Dieudonné crystal of G on the PD-thickening $S \to \mathcal{O}_K$ (which sends all divided powers of E to zero).

Let G be a p-divisible group over $\mathcal{O}_K$ with absolute filtered prismatic Dieudonné crystal $\underline{\mathcal{M}}_{\Delta}(G)$. We use Lemma 4.2.4 and Proposition 4.1.4 and consider $\underline{\mathcal{M}}_{\Delta}(G)$ as a crystal on the absolute prismatic site $(\mathcal{O}_K)_{\Delta}$.

Lemma 5.2.11. *There is a natural Frobenius equivariant, filtered isomorphism*

$$\alpha_K \colon \mathcal{M}_{\Delta}(G)(S,(p)) \xrightarrow{\simeq} \mathbb{D}(G)(S).$$

Here $\mathbb{D}(G)(S)$ denotes the evaluation of the Dieudonné crystal of G at the PD-thickening $S \to \mathcal{O}_K$.

Proof. This follows from Lemma 4.3.4. □

We want to show that the natural isomorphism α_K restricts to an isomorphism $\mathcal{M}_{\Delta}(G)((\mathfrak{S},(E)) \cong M^{\mathrm{Kis}}(G)$. In other words, we want to prove the existence of the dotted morphisms in the diagram

$$\begin{tikzcd} \mathcal{M}_{\Delta}(G)((\mathfrak{S},(E)) \arrow[r, dashed, bend left] \arrow[d, hook] & M^{\mathrm{Kis}}(G) \arrow[l, dashed, bend left] \arrow[d, hook] \\ \mathcal{M}_{\Delta}(G)(S,(p)) \arrow[r, equal, "\sim"] & \mathbb{D}(G)(S). \end{tikzcd}$$

Let C be the completion of an algebraic closure of K and let $\mathcal{O}_C \subseteq C$ be its ring of integers. Set $A_{\mathrm{inf}} := A_{\mathrm{inf}}(\mathcal{O}_C)$, $A_{\mathrm{crys}} := A_{\mathrm{crys}}(\mathcal{O}_C)$.

We can extend the morphism $\mathcal{O}_K \to \mathcal{O}_C$ to a morphism of prisms[46]

$$f : (\mathfrak{S},(E)) \to (A_{\mathrm{inf}},(\xi))$$

by sending $u \mapsto \pi^\flat = [(\pi, \pi^{1/p}, \ldots)]$ (after choosing a compatible system of p-power roots $\pi^{1/p^n} \in \mathcal{O}_C$ of π). Let

$$\psi_C \colon A_{\mathrm{inf}} \xrightarrow{\varphi} A_{\mathrm{inf}} \to A_{\mathrm{crys}}.$$

Then analogous ψ_C induces a morphism $(A_{\mathrm{inf}},(\xi)) \to (A_{\mathrm{crys}},(p))$ of prisms.

By faithful flatness of $\mathfrak{S} \to A_{\mathrm{inf}}$ (cf. [10, Lemma 4.30][47]) it suffices to prove the existence of the dotted arrows after base change to A_{inf}:

(4)
$$\begin{tikzcd} \mathcal{M}_{\Delta}(G)((\mathfrak{S},(E)) \otimes_{\mathfrak{S},f} A_{\mathrm{inf}} \arrow[r, dashed, bend left] \arrow[d, hook] & M^{\mathrm{Kis}}(G) \otimes_{\mathfrak{S},f} A_{\mathrm{inf}} \arrow[l, dashed, bend left] \arrow[d, hook] \\ \mathcal{M}_{\Delta}(G)(S,(p)) \otimes_{\mathfrak{S},f} A_{\mathrm{inf}} \arrow[r, equal, "\sim"] & \mathbb{D}(G)(S) \otimes_{\mathfrak{S},f} A_{\mathrm{inf}}. \end{tikzcd}$$

By flat base change of PD-envelopes (cf. [49, Tag 07HD]) we get

$$S \hat{\otimes}_{\mathfrak{S}} A_{\mathrm{inf}} \cong A_{\mathrm{crys}}$$

and thus $\mathbb{D}(G)(S) \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \cong \mathbb{D}(G_{\mathcal{O}_C})(A_{\mathrm{crys}})$.

Similar to Lemma 5.2.11 there is a canonical isomorphism

$$\alpha_C \colon \mathcal{M}_{\Delta}(G)((A_{\mathrm{crys}},(p)) \cong \mathbb{D}(G_{\mathcal{O}_C})(A_{\mathrm{crys}})$$

by Lemma 4.3.4 and thus the lower horizontal isomorphism in (Equation (4)) identifies with α_C. By the crystal property of $\mathcal{M}_{\Delta}(G)$ the left vertical injection

$$\mathcal{M}_{\Delta}(G)((\mathfrak{S},(E))) \otimes_{\mathfrak{S},f} A_{\mathrm{inf}} \hookrightarrow \mathcal{M}_{\Delta}(G_{\mathcal{O}_C})(S,(p)) \otimes_{\mathfrak{S},f} A_{\mathrm{inf}}.$$

identifies with the inclusion

$$\mathcal{M}_{\Delta}(G)((A_{\mathrm{inf}},(\xi))) \hookrightarrow \mathcal{M}_{\Delta}(G_{\mathcal{O}_C})(A_{\mathrm{crys}},(p))$$

[46]Note that we take ξ, not $\tilde{\xi}$.

[47]But note that our map f differs from the one of [10], which is $\varphi \circ f$.

along the morphisms of prism $\psi_C\colon (A_{\mathrm{inf}},(\xi)) \to (A_{\mathrm{crys}},(p))$. By Proposition 4.3.7 there is a canonical isomorphism

$$\beta\colon \varphi^*_{A_{\mathrm{inf}}}\mathcal{M}_{\Delta}(G)((A_{\mathrm{inf}},(\xi))) = \mathcal{M}_{\Delta}(G)((A_{\mathrm{inf}},(\tilde{\xi}))) \cong M^{\mathrm{SW}}(G_{\mathcal{O}_C})^*$$

to the dual of the functor constructed by Scholze-Weinstein. By [47, Theorem 14.4.3.] $M^{\mathrm{SW}}(G)^*\otimes_{A_{\mathrm{inf}}} A_{\mathrm{crys}} \cong \mathbb{D}(G_{\mathcal{O}_C/p})(A_{\mathrm{crys}})$ and moreover the diagram

$$\begin{tikzcd}
\varphi^*_{A_{\mathrm{inf}}}\mathcal{M}_{\Delta}(G)((A_{\mathrm{inf}},(\xi))) \arrow[r, "\beta"] \arrow[d] & M^{\mathrm{SW}}(G_{\mathcal{O}_C})^* \arrow[d] \\
\mathcal{M}_{\Delta}(G)((A_{\mathrm{crys}},(p))) \arrow[r, "\simeq"] & \mathbb{D}(G)(A_{\mathrm{crys}}) \cong M^{\mathrm{SW}}(G_{\mathcal{O}_C})^*\otimes_{A_{\mathrm{inf}}} A_{\mathrm{crys}}
\end{tikzcd}$$

commutes. Hence, it suffices to prove that there exists an isomorphism

$$\gamma\colon M^{\mathrm{Kis}}(G)\otimes_{\mathfrak{S},g} A_{\mathrm{inf}} \to M^{\mathrm{SW}}(G_{\mathcal{O}_C})^*$$

where $g=\varphi\circ f$ is a morphism of prisms

$$g : (\mathfrak{S},(E)) \to (A_{\mathrm{inf}},(\tilde{\xi})),$$

such that the diagram

$$\begin{tikzcd}
M^{\mathrm{Kis}}(G)\otimes_{\mathfrak{S},g} A_{\mathrm{inf}} \arrow[r, "\gamma"] \arrow[d] & M^{\mathrm{SW}}(G_{\mathcal{O}_C})^* \arrow[d] \\
\mathbb{D}(G_{\mathcal{O}_C})(A_{\mathrm{crys}},(p)) \arrow[r, "\simeq"] & M^{\mathrm{SW}}(G_{\mathcal{O}_C})^*\otimes_{A_{\mathrm{inf}}} A_{\mathrm{crys}}
\end{tikzcd}$$

commutes.

Let T be the dual of the p-adic Tate module T_pG of G. Then T is a lattice in a crystalline representation of $\mathrm{Gal}(\overline{K}/K)$ (where $\overline{K}\subseteq C$ is the algebraic closure of K) and $M^{\mathrm{Kis}}(G)\cong M(T)$ where $M(-)$ is Kisin's functor from lattices in crystalline representations to Breuil-Kisin modules. By [10, Proposition 4.34] $M(T)\otimes_{\mathfrak{S},g} A_{\mathrm{inf}}$ corresponds under Fargues' equivalence (cf. [47, Theorem 14.1.1]) to the pair (T,Ξ) with $\Xi\subseteq T\otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ the B^+_{dR}-lattice generated by $D_{\mathrm{dR}}(T_{\mathbb{Q}_p}) = (T\otimes_{\mathbb{Z}_p} B_{\mathrm{dR}})^{\mathrm{Gal}(\overline{K}/K)}$. But this pair is exactly the one associated to $G_{\mathcal{O}_C}$ by Scholze-Weinstein.

Thus in the end our discussion implies the following proposition.

Proposition 5.2.12. *The two functors*

$$G \mapsto M^{\mathrm{Kis}}(G)$$
$$G \mapsto \mathcal{M}_{\Delta}(G)(\mathfrak{S},(E))$$

from p-divisible groups over $\mathcal{O}_K$ to minuscule Breuil-Kisin modules are naturally isomorphic.

5.3. **Filtered prismatic Dieudonné crystals and displays.** The work of Zink provides a classification of *connected* p-divisible groups over p-adically complete rings (cf. [50]). In this section, we want to relate it to the classification obtained (for quasi-syntomic rings) in Theorem 4.9.5.

Definition 5.3.1. Let R be a p-complete ring. A *display* over R is a window (cf. Section 4.1 and [37, Example 5.4.]) over the frame

$$\underline{W}(\mathcal{O}) = (W(\mathcal{O}), I(\mathcal{O}) := \ker(W(\mathcal{O})\to\mathcal{O}), F, F_1),$$

in the topos of sheaves on the p-completely faithfully flat site of R, where F is the Witt vector Frobenius and F_1 the inverse of the bijective Verschiebung morphism $V\colon I(\mathcal{O}) \to W(\mathcal{O})$.

The category of displays over R is denoted by $\mathrm{Disp}(R)$.

Remark 5.3.2. The category of displays satisfies faithfully flat descent : see [50, Theorem 37]. Since displays over a p-complete ring R (with bounded p^∞-torsion) are equivalent to compatible systems of displays over R/p^n for all $n \geq 1$, we see that displays even satisfy p-completely faithfully flat descent (cf. [11, Corollary 4.8]). Hence the category of displays over R in the sense of Definition 5.3.1 is the same as the usual category of displays over R (i.e., windows over the frame $(W(R), I(R), F, F_1)$).

Proposition 5.3.3. *Let R be a quasi-regular semiperfectoid ring. Assume that $pR = 0$ or that R is p-torsion free. The natural morphism from Theorem 3.4.6*

$$\Delta_R \to R$$

(given by moding out $\mathcal{N}^{\geq 1}\Delta_R$) lifts to a morphism of frames (in the general sense of [16, Definition 2.1.5]*)*

$$\underline{\Delta}_{R,\mathrm{Nyg}} \to \underline{W}(R),$$

where $\underline{\Delta}_{R,\mathrm{Nyg}}$ is the frame associated to (Δ_R, I) and $\tilde{\xi}$, as in Example 4.1.18.

Proof. By adjunction (cf. [26, Theoreme 4]), the morphism $\Delta_R \to R$ gives rise to a morphism of δ-rings :

$$f\colon \Delta_R \to W(R),$$

lifting the morphism to R, i.e., sending $\mathcal{N}^{\geq 1}\Delta_R$ to $I(R)$. In particular, $f(\xi) \in I(R)$, and thus

$$f(\tilde{\xi}) = \varphi(f(\xi)) = p\varphi_1(f(\xi))$$

and so p divides $f(\tilde{\xi})$. By [12, Lemma 2.24], we deduce that $(p) = (f(\tilde{\xi}))$. It is then easy to conclude when $W(R)$ is p-torsion free since the commutation (up to a unit) of f with the divided Frobenius can be proved after multiplying by p. In the case where $pR = 0$ one argues as in [36, Lemma 7.4]. □

It would be nice to prove that for any R quasi-regular semiperfectoid, the morphism of the proposition always defines a morphism of frames. Although we did not succeed in doing so, the next proposition shows that one can circumvent this difficulty.

Proposition 5.3.4. *Let R be a quasi-syntomic ring. There exists a natural functor, unique up to isomorphism,*

$$\underline{Z}_R : \mathrm{BT}(R) \to \mathrm{Disp}(R),\ G \mapsto \underline{Z}_R(G) = (Z_R(G), \mathrm{Fil}\ Z_R(G), F, F_1)$$

such that the triple $(Z_R(G), \mathrm{Fil}\ Z_R(G), F)$ is obtained by base change of $\underline{\mathcal{M}}_{\Delta}(G)$ along the morphism of δ-pairs

$$\left(\mathcal{O}^{\mathrm{pris}}, \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}\right) \to \left(W(\mathcal{O}), \ker(W(\mathcal{O}) \to \mathcal{O})\right).$$

Moreover, it coincides with the composition of the filtered prismatic Dieudonné functor with the functor induced by the morphism of frames of Proposition 5.3.3 when R is quasi-regular semiperfectoid and $pR = 0$ or R is p-torsion free.

Proof. The requirement of the proposition already says what

$$(Z_R(G), \mathrm{Fil}\ Z_R(G), F)$$

must be. Therefore, the only issue is to define the divided Frobenius F_1.

Assume first that R is quasi-regular semiperfectoid and p-torsion free. Then one defines $\underline{Z}_R$ as the composition of the filtered prismatic Dieudonné functor with the functor induced by the morphism of frames of Proposition 5.3.3. By quasi-syntomic descent (Remark 5.3.2), one gets a functor Z_R for any p-torsion free quasi-syntomic ring R. For such rings R, the functor $\underline{Z}_R$ is necessarily unique by p-torsion freeness of $W(R)$. In particular, it commutes with base change in R.

To obtain the functor Z_R in general, we use smoothness of the stack of p-divisible groups, following an idea of Lau, [34, Proposition 2.1]. Let $X = \mathrm{Spec}(A) \to \mathcal{BT} \times \mathrm{Spec}(\mathbb{Z}_p)$ be a smooth presentation of the stack of p-divisible groups as in loc. cit. Then $\mathrm{Spec}(B) = X \times_{\mathcal{BT}} X$ is affine. The p-adic completions $\hat{A}$ and $\hat{B}$ are both p-torsion free (cf. [34, Lemma 1.6.]).

Let R be a quasi-syntomic ring and G be a p-divisible group over R. It gives rise to a map $\alpha : \mathrm{Spec}(R) \to \mathcal{BT} \times \mathrm{Spec}(\mathbb{Z}_p)$. Let

$$\mathrm{Spec}(S) = \mathrm{Spec}(R) \times_{\mathcal{BT} \times \mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(A),$$

and

$$\mathrm{Spec}(T) = \mathrm{Spec}(S) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B).$$

Let $\hat{S}$ and $\hat{T}$ be their p-adic completions. The rings $\hat{A}$ and $\hat{B}$ are p-completely smooth, and therefore quasi-syntomic. By base change the rings $\hat{S}$ and $\hat{T}$ are also quasi-syntomic. The base change

$$(Z_{\hat{S}}(G_{\hat{S}}), \mathrm{Fil}\ Z_{\hat{S}}(G_{\hat{S}}), F)$$

of the triple $(Z_R(G), \mathrm{Fil}\ Z_R(G), F)$ along $R \to \hat{S}$ is also the base change of the triple

$$Z_{\hat{A}}(H_{\hat{A}}), \mathrm{Fil}\ Z_{\hat{A}}(H_{\hat{A}}), F)$$

along $\alpha \otimes \hat{A}$ of the universal p-divisible group H over A. The divided Frobenius F_1 on $Z_{\hat{A}}(H_{\hat{A}})$ (coming from the first part of the proof) therefore induces an operator F_1 on $Z_{\hat{S}}(G_{\hat{S}})$. This operator F_1 is compatible with the descent datum for the base change along the two natural maps $\hat{S} \to \hat{T}$, since the functor $Z_{\hat{B}}$ exists and is unique. By descent (Remark 5.3.2), this defines a display structure $\underline{Z}_R(G)$ on the triple $(Z_R(G), \mathrm{Fil}\ Z_R(G), F)$.

This display structure is uniquely determined by the requirement that it is compatible with the maps $R \to \hat{S}$, $\hat{S} \to \hat{A}$. In particular, it has to coincide with the composition of the filtered prismatic Dieudonné functor with the functor induced by the morphism of frames of Proposition 5.3.3 also when R is quasi-regular semiperfectoid and killed by p. □

The functor of Proposition 5.3.4 is not an antiequivalence when $p = 2$. Nevertheless, one has the following positive result, reproving the main result of [50], [32] in the special case of quasi-syntomic rings.

Proposition 5.3.5. *Let R be a quasi-syntomic ring, flat over $\mathbb{Z}/p^n$ (for some $n > 0$) or $\mathbb{Z}_p$. The functor $\underline{Z}_R$ restricts to an antiequivalence*

$$\mathrm{BT}_f(R) \cong \mathrm{Disp}_{\mathrm{nilp}}(R)$$

between the category of formal p-divisible groups over R and the category of F-nilpotent displays over R.

Recall that a display is said to be *F-nilpotent* if its Frobenius is nilpotent modulo p.

Proof. Assume first that R is quasi-regular semiperfect. The functor $\underline{Z}_R$ is the composite of the filtered prismatic Dieudonné functor, which is an antiequivalence by Theorem 4.6.9, and of the functor induced by the morphism of frames

$$(\mathbb{\Delta}_R \cong A_{\mathrm{crys}}(R), \mathcal{N}^{\geq 1}\mathbb{\Delta}_R, \varphi, \varphi_1) \to (W(R), I(R), F, F_1).$$

The morphism $\mathbb{\Delta}_R \to W(R)$ is surjective (since it is so modulo p and both sides are p-complete ; note that we assume $pR = 0$). The kernel of this morphism is generated by the elements $[x]^{(n)}$, for $x \in \ker(R^\flat \to R)$ and $n \geq 1$. For such an element, one has

$$\varphi_1([x]^{(n)}) = \frac{(np)!}{n!p}[x]^{(np)}.$$

Iterating, one sees that φ_1 is topologically nilpotent on the kernel (with respect to the p-adic topology). By Lemma 4.1.27, the functor

$$\mathrm{DF}(R) \to \mathrm{Disp}(R)$$

is an equivalence. It is easily seen that it restricts to an antiequivalence between formal p-divisible groups and F-nilpotent displays.

By quasi-syntomic descent, this yields the statement of the proposition when R is quasi-syntomic with $pR = 0$. In general, R/p is quasi-syntomic ([11, Lemma 4.16 (2)]) and one can consider the following commutative diagram :

$$\begin{array}{ccc} \mathrm{BT}(R) & \xrightarrow{\underline{Z}_R} & \mathrm{Disp}(R) \\ \downarrow & & \downarrow \\ \mathrm{BT}(R/p) & \xrightarrow{\underline{Z}_{R/p}} & \mathrm{Disp}(R/p). \end{array}$$

Grothendieck-Messing theory for F-nilpotent displays (cf. [50, Theorem 48]) coupled with Grothendieck-Messing theory for p-divisible groups (cf. [42, V (1.6)] and [50, Corollary 97]) show that this diagram is 2-cartesian. Since $\underline{Z}_{R/p}$ is an antiequivalence, $\underline{Z}_R$ also is one. $\square$

5.4. **Étale comparison for p-divisible groups.** Let R be a quasi-syntomic ring and let G be a p-divisible group over R. In this section we show how the (dual of the) Tate module of the generic fiber of R, seen as a diamond ([46, Definition 11.1]), can be recovered from the (filtered) prismatic Dieudonné crystal $\mathcal{M}_{\mathbb{\Delta}}(G)$ of G.

Let

$$\mathcal{O}^{\mathrm{pris}}$$

be the prismatic sheaf on $(R)_{\mathrm{qsyn}}$ and

$$\mathcal{I} := \mathcal{I}^{\mathrm{pris}} \subseteq \mathcal{O}^{\mathrm{pris}}$$

the natural invertible $\mathcal{O}^{\mathrm{pris}}$-module (cf. Definition 4.1.1). Fix $n \geq 0$. Note that the Frobenius

$$\varphi\colon \mathcal{O}^{\mathrm{pris}} \to \mathcal{O}^{\mathrm{pris}}$$

induces a morphism, again called Frobenius,

$$\varphi\colon \mathcal{O}^{\mathrm{pris}}/p^n[1/\mathcal{I}] \to \mathcal{O}^{\mathrm{pris}}/p^n[1/\mathcal{I}]$$

as $\varphi(\mathcal{I}) \subseteq (p, \mathcal{I})$ although $\mathcal{I}$ is not stable under φ.

We let

$$(R)_v$$

be the v-site of all maps $\mathrm{Spf}(S) \to \mathrm{Spf}(R)$ with S a perfectoid ring over R. By definition the coverings in $(R)_v$ are v-covers $\mathrm{Spf}(S') \to \mathrm{Spf}(S)$ (cf. [12, Section 8.1.]). Let

$$(R)_{\mathrm{qsyn,qrsp}}$$

be the site of all maps $\mathrm{Spf}(S) \to \mathrm{Spf}(R)$ with S quasi-regular semiperfectoid (covers given by quasisyntomic covers). The perfectoidization functor

$$S \mapsto S_{\mathrm{perfd}}$$

from [12, Definition 8.2.] induces a continuous functor

$$\alpha\colon (R)_v \to (R)_{\mathrm{qsyn,qrsp}}$$

sending $\mathrm{Spf}(S)$ to $\mathrm{Spf}(S_{\mathrm{perfd}})$. Indeed, by [12, Proposition 8.10.] and the fact that quasi-syntomic covers are v-covers the conditions of [49, Tag 00WV] are satisfied. Moreover, we have the "inclusion of the generic fiber"

$$j\colon \mathrm{Spa}(R[1/p], R)_v^{\diamond} \to (R)_v$$

induced by sending $\mathrm{Spf}(S)$ to $\mathrm{Spa}(S[1/p], S)$[48]. Here $\mathrm{Spa}(R[1/p], R)_v^{\diamond}$ is the v-site of the diamond associated with $\mathrm{Spa}(R[1/p], R)$ (cf. [46, Section 15.1.], [46, Definition 14.1.iii)]).

Lemma 5.4.1. *There are natural isomorphism*

$$\alpha_*(\mathbb{Z}/p^n) \cong (\mathcal{O}^{\mathrm{pris}}/p^n)^{\varphi=1}$$

and

$$(\alpha \circ j)_*(\mathbb{Z}/p^n) \cong (\mathcal{O}^{\mathrm{pris}}/p^n[1/\mathcal{I}])^{\varphi=1}$$

of sheaves on $(R)_{\mathrm{qsyn,qrsp}}$.

Here $(-)^{\varphi=1}$ denotes the (non-derived) invariants of φ on the sheaf $\mathcal{O}^{\mathrm{pris}}/p^n[1/\mathcal{I}]$.

Proof. We only prove the second statement. The first is similar (but easier). Let S be a quasi-regular semiperfectoid R-algebra. Then

$$(\mathcal{O}^{\mathrm{pris}}/p^n[1/\mathcal{I}])^{\varphi=1}(S) \cong (\varinjlim_{\varphi} \mathcal{O}^{\mathrm{pris}}/p^n[1/\mathcal{I}])^{\varphi=1}(S) \cong (\varinjlim_{\varphi} \mathcal{O}^{\mathrm{pris}}(S)/p^n)^{\varphi=1}.$$

The first isomorphisms follows from commuting a kernel with a filtered limit and the second as S is quasi-regular semiperfectoid (which implies that the p-torsion free sheaf $\varinjlim_{\varphi} \mathcal{O}^{\mathrm{pris}}$ has no higher cohomology over S). Then [12, Lemma 9.2.] implies that

$$(\varinjlim_{\varphi} \mathcal{O}^{\mathrm{pris}}(S)/p^n)^{\varphi=1} \cong (A_{\mathrm{inf}}(S_{\mathrm{perfd}})/p^n[1/\mathcal{I}])^{\varphi=1}.$$

[48] We use the notation $\mathrm{Spa}(S[1/p], S)$ when S is not necessarily integrally closed in $S[1/p]$.

By [12, Lemma 9.3.], the equivalence of underlying topological spaces under tilting of perfectoid spaces, [47, Theorem 7.1.1.], and [28, Proposition 3.2.7.] the right-hand side becomes

$$W_n((S_{\mathrm{perfd}}[1/p])^\flat)^{\varphi=1} \cong \mathrm{Hom}_{\mathrm{cts}}(\pi_0(\mathrm{Spa}(S_{\mathrm{perfd}}[1/p], S_{\mathrm{perfd}})), \mathbb{Z}/p^n),$$

which agrees with

$$(\alpha \circ j)_*(\mathbb{Z}/p^n)(S).$$

This finishes the proof. □

We can derive the following description of the Tate module of the generic fiber.

Proposition 5.4.2. *Let G be a p-divisible group over R with prismatic Dieudonné crystal $\mathcal{M}_{\Delta}(G)$ and let $n \geq 0$. Then*

$$j^*\alpha^*(\mathcal{M}_{\Delta}(G)/p^n[1/\mathcal{I}]^{\varphi=1})$$

is canonically isomorphic to $\mathcal{H}om_{\mathbb{Z}/p^n}(G[p^n]_\eta, \mathbb{Z}/p^n)$ where $G[p^n]_\eta$ denotes the sheaf $\mathrm{Spa}(S[1/p], S) \mapsto G[p^n](S[1/p])$ *on* $\mathrm{Spa}(R[1/p], R)^\diamond_v$.

Proof. Set $\mathcal{M} := \mathcal{M}_{\Delta}(G)$. By Lemma 4.2.6

$$\mathcal{M} \cong \mathcal{H}om_{(R)_{\mathrm{qsyn,qrsp}}}(T_pG, \mathcal{O}^{\mathrm{pris}}).$$

From the proof of Proposition 4.6.5 we can conclude that

$$\begin{aligned}\mathcal{H}om_{(R)_{\mathrm{qsyn,qrsp}}}(T_pG, \mathcal{O}^{\mathrm{pris}})/p^n &\cong \mathcal{H}om_{(R)_{\mathrm{qsyn,qrsp}}}(T_pG, \mathcal{O}^{\mathrm{pris}}/p^n) \\ &\cong \mathcal{H}om_{(R)_{\mathrm{qsyn,qrsp}}}(G[p^n], \mathcal{O}^{\mathrm{pris}}/p^n).\end{aligned}$$

It follows that

$$\mathcal{M}/p^n[1/\mathcal{I}] \cong \mathcal{H}om_{(R)_{\mathrm{qsyn,qrsp}}}(G[p^n], \mathcal{O}^{\mathrm{pris}}/p^n[1/\mathcal{I}])$$

as using Section 4.4 the functor $\mathcal{H}om_{(R)_{\mathrm{qsyn,qrsp}}}(G[p^n], -)$ commutes with filtered colimits. Finally,

$$\mathcal{M}/p^n[1/\mathcal{I}]^{\varphi=1} \cong \mathcal{H}om_{(R)_{\mathrm{qsyn,qrsp}}}(G[p^n], \mathcal{O}^{\mathrm{pris}}/p^n[1/\mathcal{I}]^{\varphi=1}).$$

By Lemma 5.4.1

$$\mathcal{O}^{\mathrm{pris}}/p^n[1/\mathcal{I}]^{\varphi=1} \cong (\alpha \circ j)_*(\mathbb{Z}/p^n)$$

and thus

$$\begin{aligned}\mathcal{M}/p^n[1/\mathcal{I}]^{\varphi=1} &\cong \mathcal{H}om_{(R)_{\mathrm{qsyn,qrsp}}}(G[p^n], (\alpha \circ j)_*(\mathbb{Z}/p^n)) \\ &\cong (\alpha \circ j)_*(\mathcal{H}om_{\mathbb{Z}/p^n}((\alpha \circ j)^*G[p^n], \mathbb{Z}/p^n))).\end{aligned}$$

The definitions of α and j imply

$$(\alpha \circ j)^*\mathcal{M}/p^n[1/\mathcal{I}]^{\varphi=1} \cong \mathcal{H}om_{\mathbb{Z}/p^n}((\alpha \circ j)^*G[p^n], \mathbb{Z}/p^n),$$

as we can conclude from Lemma 5.4.3. □

Lemma 5.4.3. *With the notations from Proposition 5.4.2,*

$$(\alpha \circ j)^*G[p^n] \cong G[p^n]_\eta.$$

Proof. By right exactness of $(\alpha \circ j)^*$, it suffices to show

$$(\alpha \circ j)^* T_p G \cong T_p G_\eta.$$

Moreover, we may assume that R is perfectoid by passing to slice topoi. Let S be the R-algebra representing T_pG on p-complete rings. Thus S is the p-completion of $\varinjlim_m S_m$ where S_m represents $G[p^m]$. Then S is quasi-regular semiperfectoid. By definition $(\alpha \circ j)^* T_p G$ is represented by the perfectoid space

$$\mathrm{Spa}(S_{\mathrm{perfd}}[1/p], S^+_{\mathrm{perfd}})$$

over $\mathrm{Spa}(R[1/p], R)$ where S^+_{perfd} is the integral closure of S_{perfd} in $S_{\mathrm{perfd}}[1/p]$. Let $\mathrm{Spa}(T, T^+)$ be an affinoid perfectoid space over $\mathrm{Spa}(R[1/p], R)$, in particular we assume that T^+ is integrally closed in $T = T^+[1/p]$. Then any morphism $S_{\mathrm{perfd}}[1/p] \to T$ sends $S^+_{\mathrm{perfd}} \to T^+$ because S is a p-completed direct limit of finite R-algebras and T^+ is perfectoid and integrally closed in T. Thus

$$\begin{aligned} \mathrm{Hom}_{(R[1/p],R)}((S_{\mathrm{perfd}}[1/p], S^+_{\mathrm{perfd}}), (T, T^+)) &\cong \mathrm{Hom}_R(S^+_{\mathrm{perfd}}, T^+) \\ &\cong \mathrm{Hom}_R(S, T^+) \\ &\cong \mathrm{Hom}_R(\varinjlim_m S_m, T^+) \\ &\cong \mathrm{Hom}_R(\varinjlim_m S_m, T) = T_p G(T) \end{aligned}$$

where S_m represents $G[p^m]$ (thus S is the p-adic completion of $\varinjlim_m S_m$)). In the last isomorphism we used again that all S_m are finite over R and thus any morphism $S_m \to T$ of R-algebras factors over T^+. $\square$

Appendix A. Descent for p-completely faithfully flat morphisms

In this appendix, we prove descent results for the p-completely faithfully flat topology on p-complete rings. We expect the statements in this section to be more or less known, but we did not find a written source.

Let R be a p-complete ring with bounded p^∞-torsion. The property that R has bounded p^∞-torsion is important as it implies that the pro-systems

$$\{R/p^n\}_{n\geq 0}$$

and

$$\{R\otimes^{\mathbb{L}}_{\mathbb{Z}}\mathbb{Z}/p^n\}_{n\geq 0}$$

are pro-isomorphic[49]. In particular, for every complex M of R-modules there is an isomorphism of pro-systems

$$\{M\otimes^{\mathbb{L}}_{R}R/p^n\}_{n\geq 0}\cong\{M\otimes^{\mathbb{L}}_{\mathbb{Z}}\mathbb{Z}/p^n\}_{n\geq 0}.$$

Let us recall how faithfully flat descent for modules follows from the Barr-Beck theorem (cf. [20, Chapitre 4]).

Let

$$T\colon\mathcal{C}\to\mathcal{D}$$

be a functor between two categories and assume that T admits a rightadjoint

$$U\colon\mathcal{D}\to\mathcal{C}.$$

Set

$$F:=T\circ U\colon\mathcal{D}\to\mathcal{D}.$$

The unit $c\colon F=TU\to\mathrm{Id}_{\mathcal{D}}$ and the natural transformation

$$\mu\colon F=TU\to F\circ F=TUTU$$

induced from the counit $\mathrm{Id}_{\mathcal{C}}\to UT$ give F the structure of a comonad. An object $D\in\mathcal{D}$ together with a morphism $\alpha_D\colon D\to FD$ is called an comodule for F if the composition

$$D\xrightarrow{\alpha_D}FD\xrightarrow{c}D$$

is the identity and the diagram

$$\begin{tikzcd} D \arrow[r, "\alpha_D"] \arrow[d] & FD \arrow[d, "\mu_D"] \\ FD \arrow[r, "F\alpha_D"] & FFD \end{tikzcd}$$

commutes. For each $C\in\mathcal{C}$ the canonical morphism

$$TC\to TUTC$$

induced from the counit $C\to TUC$ gives a comodule structure on the object $D:=TC$. The theorem of Barr-Beck states the following converse.

Theorem A.1 (Barr-Beck)**.** *Assume that*

- *A pair of morphisms $a,b\colon C_1\to C_2$ in $\mathcal{C}$ admits an equalizer if $Ta,Tb\colon TC_1\to TC_2$ admits one*
- *If $c\colon C_0\to C_1$ equalizes a,b, then c is an equalizer of a,b if Tc is an equalizer of Ta, Tb.*

[49] The pro-system $H^{-1}(R\otimes^{\mathbb{L}}_{\mathbb{Z}}\mathbb{Z}/p^n)\cong R[p^n]$ is pro-zero as R has bounded p^∞-torsion.

Then the functor T induces an equivalence of $\mathcal{C}$ with the category of comodules for F.

If $A \to B$ is a faithfully flat morphism of rings, then faithfully flat descent be deduced from Theorem A.1 by setting $\mathcal{C} := \mathrm{Mod}_A$, $\mathcal{D} := \mathrm{Mod}_B$ and $T := B \otimes_A (-)$. Namely in this case comodules in $\mathcal{D}$ can be identified with B-modules with descent datum (cf. [20, Chapitre 4.2]) and the theorem of Barr-Beck applies because the functor $T = B \otimes_A (-)$ satisfies both hypothesis in Theorem A.1 as $A \to B$ is faithfully flat.

Assume from now on that R a p-complete ring with bounded p^∞-torsion. Let $\mathrm{Mod}_{R,c}$ be the abelian category of derived p-complete R-modules and let $R \to R'$ be a p-completely flat morphism of rings and assume that R' is p-complete with bounded p^∞-torsion.

For an R-module M we set

$$T(M) := R' \widehat{\otimes^{\mathbb{L}}_R} M$$

where the completion is derived. We let

$$\mathrm{Mod}_{R,c,bdd} \subseteq \mathrm{Mod}_{R,c}$$

be the full subcategory of R-modules with bounded p^∞-torsion. Equivalently, it is the full subcategory consisting of R-modules M such that the pro-system

$$\{H^{-1}(M \otimes^{\mathbb{L}}_R R/p^n\}_{n\geq 0} \cong \{H^{-1}(M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\}_{n\geq 0}$$

is pro-zero. Namely, if M is an abelian group such that the p^∞-torsion in M is unbounded, then

$$\{H^{-1}(M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n)\}_{n\geq 0} \cong \{M[p^n]\}_{n\geq 0}$$

is not pro-zero. If it were, then there exists an $m \geq 0$, such that the morphism

$$p^{m-1} \colon M[p^m] \to M[p]$$

is zero. But as M has unbounded p^∞-torsion there exists a non-zero element in $M[p^m] \setminus M[p^{m-1}]$.

Lemma A.2. *The functor $T \colon \mathrm{Mod}_{R,c,\mathrm{bdd}} \to \mathrm{Mod}_{R',c,\mathrm{bb}}$ has the forgetful functor $\mathrm{Mod}_{R',c,\mathrm{bdd}} \to \mathrm{Mod}_{R,c,\mathrm{bdd}}$ as a right adjoint.*

Proof. The functor T is well-defined. Namely, for $M \in \mathrm{Mod}_{R,c,\mathrm{bdd}}$ the complex $T(M)$ is p-adically complete and $T(M) \otimes^{\mathbb{L}}_{R'/p} R'/p \cong M \otimes^{\mathbb{L}}_{R/p} R'/p$ is concentrated in degree 0 as $R \to R'$ is p-completely flat, and the pro-system

$$\{H^{-1}(M \otimes^{\mathbb{L}}_R R'/p^n)\}_{n\geq 0} \cong \{H^{-1}(M \otimes^{\mathbb{L}}_R R/p^n) \otimes_{R/p^n} R'/p^n\}_{n\geq 0}$$

is zero. For the adjunction it suffices to see that the restriction of a derived p-complete R'-module M with bounded p^∞-torsion is a derived p-complete R-module with bounded p^∞-torsion. But derived p-completeness and the torsion only depends on the underlying $\mathbb{Z}$-module and thus the claim follows. $\square$

Lemma A.3. *Let $0 \to M_0 \to M_1 \to M_2 \to 0$ be an exact sequence in $\mathrm{Mod}_{R,c,\mathrm{bdd}}$. Then the sequence*

$$0 \to T(M_0) \to T(M_1) \to T(M_2) \to 0$$

is exact.

Proof. The triangle

$$\widehat{M_0 \otimes^{\mathbb{L}}_R R'} \to \widehat{M_1 \otimes^{\mathbb{L}}_R R'} \to \widehat{M_2 \otimes^{\mathbb{L}}_R R'}$$

is distinguished and each term is concenctrated in degree 0. □

Lemma A.4. *The category* $\mathrm{Mod}_{R,c,\mathrm{bdd}}$ *of* p*-complete* R*-modules with bounded* p^∞*-torsion admits kernels and the inclusion* $\mathrm{Mod}_{R,c,\mathrm{bdd}} \to \mathrm{Mod}_R$ *commutes with these. Moreover, the functor* $T\colon \mathrm{Mod}_{R,c,\mathrm{bdd}} \to \mathrm{Mod}_{R',c,\mathrm{bdd}}$ *commutes with kernels.*

Proof. Let $M_1 \to M_2$ be a morphism in $\mathrm{Mod}_{R,c,\mathrm{bdd}}$ and K its kernel as a morphism of R-modules. Then K is (derived) p-complete (because it identifies with a cohomology group of the derived p-complete mapping cone of $M_1 \to M_2$). Moreover, K has bounded p^∞-torsion because M_1 has. In the exact sequence

$$0 \to K \to M_1 \to M_1/K \to 0$$

the right term is (derived) p-complete and has bounded p^∞-torsion (because it embeds into M_2). Therefore the statement on p-completely flat base change follows from Lemma A.3 □

Lemma A.5. *The functor*

$$T\colon \mathrm{Mod}_{R,c,\mathrm{bdd}} \to \mathrm{Mod}_{R',c,\mathrm{bdd}}$$

is conservative, i.e., reflects isomorphisms.

Proof. Let $M_1 \to M_2$ be a morphism, such that $T(M_1) \to T(M_2)$ is an isomorphism. By p-completeness of M_1 and M_2 it suffices to see that $M_1 \otimes^{\mathbb{L}}_R R/p \to M_2 \otimes^{\mathbb{L}}_R R/p$ is an isomorphism. By p-complete faithful flatness of $R \to R'$ this may be checked after base change to R'. But for $M \in \mathrm{Mod}_{R,c,\mathrm{bdd}}$ we get that

$$M \otimes^{\mathbb{L}}_R R/p \otimes^{\mathbb{L}}_{R/p} R'/p \cong M \otimes^{\mathbb{L}}_R R' \otimes^{\mathbb{L}}_{R'} R'/p \cong (\widehat{M \otimes^{\mathbb{L}}_R R'}) \otimes^{\mathbb{L}}_{R'} R'/p \cong T(M) \otimes^{\mathbb{L}}_{R'} R'/p$$

and thus the statement follows. □

Lemma A.6. *Let* $R \to R'$ *be* p*-completely faithfully flat and let* $0 \to M_0 \to M_1 \to M_2$ *be a sequence of morphisms in* $\mathrm{Mod}_{R,c,\mathrm{bdd}}$ *such that* $0 \to T(M_0) \to T(M_1) \to T(M_2)$ *is exact. Then* $0 \to M_0 \to M_1 \to M_2$ *is exact.*

Proof. Let K be the kernel of $M_1 \to M_2$. It suffices to see that the natural map $M_0 \to K$ is an isomorphism. Using Lemma A.4 this follows from Lemma A.3. □

Now, we can conclude descent for p-completed R-modules with bounded p^∞-torsion from the Barr-Beck theorem.

Theorem A.7. *The category* $\mathrm{Mod}_{R,c,\mathrm{bdd}}$ *is equivalent to the category of descent data* (M, α)*, where* $M \in \mathrm{Mod}_{R',c,\mathrm{bdd}}$ *and* α *an isomorphism of the base changes of* M *to* $\widehat{R' \otimes^{\mathbb{L}}_R R'}$ *satisfying the cocycle condition over* $R' \otimes^{\mathbb{L}}_R \widehat{R' \otimes^{\mathbb{L}}_R} R'$*.*

Proof. From Lemma A.6 and Lemma A.4 we can conclude that the hypothesis of the theorem of Barr-Beck Theorem A.1 are satisfied. As in [20, Chapitre 4.2.] one can conclude the statement on descent data. □

Under descent the property of being finite projective is preserved.

Lemma A.8. *Let* $M \in \mathrm{Mod}_{c,R,\mathrm{bdd}}$*. If* $T(M) = \widehat{M \otimes^{\mathbb{L}}_R R'}$ *is finite projective over* R'*, then* M *is finite projective over* R*.*

Proof. From classical faithfully flat descent of finite projective modules we can conclude that

$$M \otimes^{\mathbb{L}}_R R/p$$

is a finite projective R/p-module as

$$M \otimes^{\mathbb{L}}_R R/p \otimes^{\mathbb{L}}_{R/p} R'/p$$

is. Therefore the claim follows from Lemma A.9. □

Lemma A.9. *Let R be a p-complete ring with bounded p^∞-torsion and let M be a derived p-complete complex of R-modules such that $M \otimes^{\mathbb{L}}_R R/p$ is a finite projective R/p-module placed in degree 0. Then M is (quasi-isomorphic to) a finite projective R-module.*

Proof. We first show that M is concentrated in degree 0. Indeed,

$$M \cong R\varprojlim_n (M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n) \cong R\varprojlim_n (M \otimes^{\mathbb{L}}_R R/p^n),$$

where we use that R has bounded p^∞-torsion in the second isomorphism. Moreover, all $M \otimes^{\mathbb{L}}_R R/p^n$ are concentrated in degree 0 by the flatness of $M \otimes^{\mathbb{L}}_R R/p$ over R/p and the exact triangles

$$(M \otimes^{\mathbb{L}}_R R/p) \otimes^{\mathbb{L}}_{R/p} (p^n)/(p^{n+1}) \to M \otimes^{\mathbb{L}}_R R/p^{n+1} \to M \otimes^{\mathbb{L}}_R R/p^n.$$

By Mittag-Leffler we thus get that M is concentrated in degree 0. By p-completeness of M we can conclude from the derived Nakayama lemma that M is finitely generated because $M \otimes^{\mathbb{L}}_R R/p$ is. Therefore there exists a short exact sequence

$$0 \to K \to R^n \to M \to 0.$$

Then K is derived p-complete and because $M \otimes^{\mathbb{L}}_R R/p$ is concentrated in degree 0 the sequence

$$0 \to K/p \to (R/n)^n \to M/p \to 0$$

remains exact. As $M/p \cong M \otimes^{\mathbb{L}}_R R/p$ is projective over R/p, K/p is finitely generated. Therefore K is finitely generated by the derived Nakayama lemma and therefore M is finitely presented. Let $x \in \mathrm{Spec}(R)$ be a closed point. Then by p-completeness of R the point x lies in $\mathrm{Spec}(R/p)$. Thus we can conclude that

$$M \otimes^{\mathbb{L}}_R k(x) \cong (M \otimes^{\mathbb{L}}_R R/p) \otimes^{\mathbb{L}}_{R/p} k(x)$$

is concentrated in degree 0 as $M \otimes^{\mathbb{L}}_R R/p$ is a finite projective R/p-module. Using Lemma A.10 we can conclude. □

Lemma A.10. *Let S be a ring and let M be a finitely presented S-module. If*

$$M \otimes^{\mathbb{L}}_S k(x)$$

is concentrated in degree 0 for all closed points $x \in \mathrm{Spec}(S)$, then M is finite projective.

Proof. Using that M is finitely presented we may after replacing S by a localisation at a maximal ideal assume that S is local with unique closed point $x \in \mathrm{Spec}(S)$. Then let

$$0 \to K \to S^n \xrightarrow{\alpha} M \to 0$$

be a short exact sequence such that $\alpha \otimes_S k(x)$ is an isomorphism. Then K is finitely generated because M is finitely presented. Hence, by Nakayama it suffices to prove

$K \otimes_S k(x) = 0$. But $M \otimes_S^{\mathbb{L}} k(x)$ is discrete and $\alpha \otimes_S k(x)$ is an isomorphism, thus even $K \otimes_S^{\mathbb{L}} k(x) = 0$. $\square$

Proposition A.11. *The fibered categories of p-divisible groups and finite locally free group schemes over p-complete rings with bounded p^∞-torsion are stacks for the p-completely faithfully flat topology.*

Proof. It suffices to show the statement for finite locally free group schemes as p-divisible groups are canonically a colimit of such. From Theorem A.7 and Lemma A.8 we know that finite locally free modules form a stack for the p-completely faithfully flat topology on p-complete rings with bounded p^∞-torsion. As base change commutes with fiber products, this implies that finite locally free group schemes form a stack, too. $\square$

Recall that a morphism

$$(A, I) \to (B, J)$$

of prisms is called faithfully flat if it is (p, I)-completely flat.

Proposition A.12. *The fibered category*

$$(A, I) \mapsto \{ \textit{ finite projective } A - \textit{modules}\}$$

on the category of bounded prisms is a stack for the faithfully flat topology.

Proof. If (A, I) is a prism, then A is classically I-complete and thus finite projective A-modules are equivalent to compatible systems of finite projective A/I^n-modules, i.e.,

$$\{ \text{ finite projective } A - \text{modules}\} \cong 2 - \varprojlim_n \{ \text{ finite projective } A/I^n - \text{modules}\}$$

(cf. [49, Tag 0D4B]). As the 2-limit of stacks is again a stack it suffices to show that for any $n \geq 0$ the fibered category

$$(A, I) \mapsto \{ \text{ finite projective } A/I^n - \text{modules}\}$$

is a stack on bounded prisms. If $(A, I) \to (B, J)$ is a faithfully flat morphim of prisms, then

$$A/I^n \to B/J^n$$

is a p-completely faithfully flat morphism of rings with bounded p^∞-torsion. Thus the proposition follows from Theorem A.7 and Lemma A.8. $\square$

Example A.13. We give an example of a ring R which is classically (p, f)-complete where $f \in R$ is a non-zero divisor, such that R/f has bounded p^∞-torsion, but R has unbounded p^∞-torsion. Set

$$R := \mathbb{Z}[f, x_{i,j} |\ i \geq 0, 0 \leq j \leq i]^{\wedge (p,f)}/J$$

with J generated by the elements

$$p x_{i,j} - f x_{i,j+1}$$

(where $x_{i,i+1} := 0$). Then f is a non-zero divisor in R and all p^∞-torsion in

$$R/f \cong \mathbb{Z}[x_{i,j}]/(p x_{i,j})$$

is killed by p. But

$$p^i x_{i,0} = p^i f x_{i,1} = \ldots = f^i x_{i,i} \neq 0$$

while $p^{i+1}x_{i,0} = f^i p x_{i,i} = 0$. This shows that R has unbounded p^∞-torsion. As f is a non-zero divisor in R the $(p, f)^\infty$-torsion in R is zero.

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