THE p-COMPLETED CYCLOTOMIC TRACE IN DEGREE 2

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Abstract. We prove that for a quasi-regular semiperfectoid $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra R (in the sense of Bhatt-Morrow-Scholze), the cyclotomic trace map from the p-completed K-theory spectrum $K(R;\mathbb{Z}_p)$ of R to the topological cyclic homology $\mathrm{TC}(R;\mathbb{Z}_p)$ of R identifies on π_2 with a q-deformation of the logarithm.

Contents

1. Introduction

Fix a prime p. The aim of this paper is to concretely identify in degree 2, for a certain class of p-complete rings R, the p-completed cyclotomic trace

$$\mathrm{ctr}\colon K(R;\mathbb{Z}_p) \to \mathrm{TC}(R;\mathbb{Z}_p)$$

from the p-completed K-theory spectrum $K(R;\mathbb{Z}_p)$ of R to the topological cyclic homology $\mathrm{TC}(R;\mathbb{Z}_p)$ of R. Our main result is that on π_2 the p-completed cyclotomic trace is given by a q-logarithm

$$\log_q(x) := \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(n-1)/2} \frac{(x-1)(x-q)\cdots(x-q^{n-1})}{[n]_q},$$

which is a q-deformation of the usual logarithm (where q is a parameter which will be defined later). Before stating a precise version of the theorem, let us try to put it in context and to explain what the involved objects are.

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1.1. **K-theory and topological cyclic homology.** We start with K-theory. For any commutative ring A, Quillen defined in [20] the algebraic K-theory space $K(A)$ of A as a generalization of the Grothendieck group $K_0(A)$ of vector bundles on the scheme $\mathrm{Spec}(A)$. The (connective) K-theory spectrum $K(A)$ of a ring A is obtained by group completing[1] the $\mathbb{E}_\infty$-monoid of vector bundles on $\mathrm{Spec}(A)$ whose addition given by the direct sum. In other words, for the full K-theory one mimicks in a homotopy theoretic context the definition of $K_0(A)$ with the set of isomorphism classes of vector bundles replaced by the groupoid of vector bundles. Algebraic K-theory behaves like a cohomology theory but has the nice feature, compared to other cohomology theories, like étale cohomology, that it only depends on the category of vector bundles on the ring (rather than on the ring itself) and thus enjoys strong functoriality properties, which makes it a powerful invariant attached to A.

Unfortunately, the calculation of the homotopy groups

$$K_i(A) := \pi_i(K(A)),\ i \geq 1,$$

is in general rather untractable. There is for example a natural embedding

$$A^\times \to \pi_1(K(A)),$$

which is an isomorphism if A is local, but the higher K-groups are much more mysterious. One essential difficulty comes from the fact that K-theory, although it is a Zariski (and even Nisnevich) sheaf of spaces (cf. [25]), does not satisfy étale descent. One could remedy this by étale sheafification, but one would loose the good properties of K-theory. This lead people to look for good *approximations* of K-theory, at least after profinite completion : by this, we mean invariants, still depending only on the category of vector bundles on the underlying ring, satisfying étale descent - and therefore, easier to compute - and close enough to (completed) K-theory, at least in some range.

The work of Thomason, [24], provides a good illustration of this principle. Thomason shows that the $K(1)$-*localization* of K-theory, with respect to a prime ℓ *invertible in* A, satisfies étale descent[2] and coincides with ℓ-adically completed (for short: ℓ-adic) K-theory in high degrees. When the prime p is not invertible in A, the situation is much more subtle. For instance, a theorem of Gabber [10] shows that ℓ-adic K-theory is insensitive to replacing A by A/I if (A, I) forms an henselian pair ; in particular, the computation of ℓ-adic K-theory of henselian rings (which form a basis of the Nisnevich topology) is reduced to the computation of the ℓ-adic K-theory of fields. This is not true anymore for p-adic K-theory. Nevertheless, the recent work of Clausen-Mathew-Morrow, [7], expresses this failure in terms of another non-commutative invariant attached to A, the *topological cyclic homology* of A, whose definition will be recalled below. Topological cyclic homology is related to K-theory via the *cyclotomic trace*

$$\mathrm{ctr}\colon K(A) \to \mathrm{TC}(A).$$

[1]Cf. [17] for a discussion of homotopy-theoretic group completions and Quillen's $+$-construction.

[2]In fact, it even coincides with ℓ-adic étale K-theory on connective covers.

Clausen, Mathew and Morrow prove, extending earlier work of Dundas, Goodwillie and McCarthy [8] in the nilpotent case[3], that the cyclotomic trace induces, for any ideal $I \subseteq A$ such that the pair (A, I) is henselian, an isomorphism

$$K(A, I)/n \cong \mathrm{TC}(A, I)/n$$

from the relative K-theory

$$K(A, I)/n := \mathrm{fib}(K(A)/n \to K(A/I)/n)$$

to the relative topological cyclic homology

$$\mathrm{TC}(A, I)/n := \mathrm{fib}(\mathrm{TC}(A)/n \to \mathrm{TC}(A/I)/n),$$

for *any* integer n. This has the consequence that p-completed TC provides a good approximation of p-adic K-theory, at least for rings henselian along (p): namely, it satisfies étale descent (because topological cyclic homology does) and coincides with p-adic K-theory in high degrees. Under additional hypotheses, one can even get better results: for instance, Clausen, Mathew and Morrow prove, among other things, that the cyclotomic trace induces an isomorphism

$$K(R; \mathbb{Z}_p) \cong \tau_{\geq 0}\mathrm{TC}(R; \mathbb{Z}_p)$$

for all rings R which are henselian along (p) and such that R/p is semiperfect (i.e., such that Frobenius is surjective), cf. [7, Corollary 6.9.].

Examples for such rings are the *quasi-regular semiperfectoid rings* of [3]. A ring R is called quasi-regular semiperfectoid, if R is p-complete with bounded p^∞-torsion[4], the p-completed cotangent complex $\widehat{L_{R/\mathbb{Z}_p}}$ has p-complete Tor-amplitude in $[-1, 0]$ and there exists a surjective morphism $R' \to R$ with R' (integral) perfectoid. This class of rings is interesting as for R quasi-regular semiperfectoid the topological cyclic homology $\pi_*(\mathrm{TC}(R; \mathbb{Z}_p))$ can be computed in more concrete terms.

Let us recall the description of topological cyclic homology $\pi_*(\mathrm{TC}(R; \mathbb{Z}_p))$ from [3], which builds heavily on the foundational work of Nikolaus and Scholze [18]. For this, we need to spell some definitions. From now on, all spectra will be assumed to be p-completed. One starts with the (p-completed) topological Hochschild homology spectrum $\mathrm{THH}(R; \mathbb{Z}_p)$ of R, which is equipped with a natural $\mathbb{T} = S^1$-action and a $\mathbb{T}$-equivariant map, the cyclotomic Frobenius,

$$\varphi_{\mathrm{cycl}} \colon \mathrm{THH}(R; \mathbb{Z}_p) \to \mathrm{THH}(R; \mathbb{Z}_p)^{tC_p}$$

to the Tate fixed points of the cyclic group $C_p \subseteq \mathbb{T}$. Then one takes the homotopy fixed points, the negative topological cyclic homology,

$$\mathrm{TC}^-(R; \mathbb{Z}_p) := \mathrm{THH}(R; \mathbb{Z}_p)^{h\mathbb{T}}$$

and the Tate fixed points, the periodic topological cyclic homology,

$$\mathrm{TP}(R; \mathbb{Z}_p) := \mathrm{THH}(R; \mathbb{Z}_p)^{t\mathbb{T}}.$$

From the cyclotomic Frobenius on $\mathrm{THH}(R)$ one derives a map[5]

$$\varphi_{\mathrm{cycl}}^{h\mathbb{T}} \colon \mathrm{TC}^-(R; \mathbb{Z}_p) \to \mathrm{TP}(R; \mathbb{Z}_p).$$

[3]This is not a generalization though, since the result of Dundas-Goodwillie-McCarthy applies also to non-commutative rings and is not restricted to finite coefficients.

[4]This means that there exists $N \geq 0$ such that $R[p^\infty] = R[p^N]$. This technical condition is useful when dealing with derived completions.

[5]Here one needs [18, Lemma II.4.2.] which implies $\mathrm{TP}(R; \mathbb{Z}_p) \cong (\mathrm{THH}(R; \mathbb{Z}_p)^{tC_p})^{h\mathbb{T}}$.

Then the topological cyclic homology is defined via the fiber sequence

$$\mathrm{TC}(R;\mathbb{Z}_p) \to \mathrm{TC}^-(R;\mathbb{Z}_p) \xrightarrow{\mathrm{can}-\varphi_{\mathrm{cycl}}^{h\mathbb{T}}} \mathrm{TP}(R;\mathbb{Z}_p),$$

where $\mathrm{can}\colon \mathrm{TC}^-(R;\mathbb{Z}_p) \to \mathrm{TP}(R;\mathbb{Z}_p)$ is the canonical map from homotopy to Tate fixed points. The ring

$$\widehat{\mathbb{\Delta}}_R := \pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p)) \cong \pi_0(\mathrm{TP}(R;\mathbb{Z}_p)).$$

is p-complete, p-torsion free and the cyclotomic Frobenius $\varphi_{\mathrm{cycl}}^{h\mathbb{T}}$ induces a Frobenius lift φ on $\widehat{\mathbb{\Delta}}_R$ (cf. [4, Theorem 11.10]).

Remark 1.1. The prismatic perspective of [4] gives an alternative description of $\widehat{\mathbb{\Delta}}_R$: it is the completion with respect to the Nygaard filtration of the (derived) prismatic cohomology $\mathbb{\Delta}_R$ of R. In particular, using the theory of δ-rings, one can give, when R is a p-complete with bounded p^∞-torsion quotient of a perfectoid ring by a regular sequence a construction of $\widehat{\mathbb{\Delta}}_R$ as the Nygaard completion of a concrete prismatic envelope (cf. [4, Proposition 3.12]).

The choice of a morphism $R' \to R$ with R' perfectoid yields a distinguished element $\tilde{\xi}$ of the ring $\widehat{\mathbb{\Delta}}_R$. Using $\tilde{\xi}$ one defines the Nygaard filtration

$$\mathcal{N}^{\geq i}\widehat{\mathbb{\Delta}}_R := \varphi^{-1}((\tilde{\xi}^i))$$

on $\widehat{\mathbb{\Delta}}_R$. The graded rings $\pi_*(\mathrm{TC}^-(R;\mathbb{Z}_p))$ and $\pi_*(\mathrm{TP}(R;\mathbb{Z}_p))$ are then concentrated in even degrees and

$$\pi_{2i}(\mathrm{TC}^-(R;\mathbb{Z}_p)) \cong \mathcal{N}^{\geq i}\widehat{\mathbb{\Delta}}_R$$
$$\pi_{2i}(\mathrm{TP}(R;\mathbb{Z}_p)) \cong \widehat{\mathbb{\Delta}}_R$$

for $i \in \mathbb{Z}$ (cf. [4, Theorem 11.10]).[6] Moreover, on π_{2i} the cyclotomic Frobenius

$$\varphi_{\mathrm{cycl}}^{h\mathbb{T}}\colon \pi_{2i}(\mathrm{TC}^-(R;\mathbb{Z}_p)) \to \pi_{2i}(\mathrm{TP}(R;\mathbb{Z}_p))$$

identifies with the divided Frobenius $\frac{\varphi}{\tilde{\xi}^i}$. Thus from the definition of $\mathrm{TC}(R)$ we obtain exact sequences

$$0 \to \pi_{2i}(\mathrm{TC}(R;\mathbb{Z}_p)) \cong \widehat{\mathbb{\Delta}}_R^{\varphi=\tilde{\xi}^i} \to \mathcal{N}^{\geq i}\widehat{\mathbb{\Delta}}_R \xrightarrow{1-\frac{\varphi}{\tilde{\xi}^i}} \widehat{\mathbb{\Delta}}_R \to \pi_{2i-1}(\mathrm{TC}(R;\mathbb{Z}_p)) \to 0.$$

As mentioned in Remark 1.1, the ring $\widehat{\mathbb{\Delta}}_R$ tends to be computable. For example, if R is perfectoid, then $\widehat{\mathbb{\Delta}}_R \cong A_{\mathrm{inf}}(R)$ is Fontaine's construction applied to R and if $pR = 0$, then $\widehat{\mathbb{\Delta}}_R$ is the Nygaard completion of the universal PD-thickening $A_{\mathrm{crys}}(R)$ of R. Thus, for quasi-regular semiperfectoid rings the target of the cyclotomic trace is rather explicit.

1.2. **Main results.** The results of [7] (together with those of [3]) therefore give a way of computing higher p-completed K-groups of quasi-regular semiperfectoid rings. But there is at least one degree (except 0) where one can be more explicit, without using the cyclotomic trace map: namely, after p-completion of $K(R)$ there is a canonical morphism

$$T_p(R^\times) \to \pi_2(K(R;\mathbb{Z}_p))$$

[6] These identifications depend on the choice of a suitable generator $v \in \pi_{-2}(\mathrm{TC}^-(R;\mathbb{Z}_p))$. If R is an algebra over $\mathbb{Z}_p^{\mathrm{cycl}}$ we will clarify our choice in Section 6 carefully.

from the Tate module $T_p(R^\times)$ of the units of R, which is an isomorphism in many cases. The results explained in the previous paragraph show that the cyclotomic trace identifies $\pi_2(K(R;\mathbb{Z}_p))$ with

$$\pi_2(\mathrm{TC}(R;\mathbb{Z}_p)) \cong \widehat{\mathbb{\Delta}}_R^{\varphi=\tilde{\xi}}.$$

What does the composite map

$$T_p(R^\times) \to \pi_2(K(R;\mathbb{Z}_p)) \xrightarrow{\mathrm{ctr}} \pi_2(\mathrm{TC}(R;\mathbb{Z}_p)) \cong \widehat{\mathbb{\Delta}}_R^{\varphi=\tilde{\xi}}$$

look like ? The main result of this paper, which we now state, provides a concrete description of it. Let R be a quasi-regular semiperfectoid ring which admits a compatible system of morphisms $\mathbb{Z}[\zeta_{p^n}] \to R$ for $n \geq 0$. These morphisms give rise to the elements

$$\varepsilon = (1, \zeta_p, \ldots) \in R^\flat = \varprojlim_{x \mapsto x^p} R\ ,\ q := [\varepsilon]_\theta \in \widehat{\mathbb{\Delta}}_R$$

and

$$\tilde{\xi} := \frac{q^p - 1}{q - 1}.$$

Here

$$[-]_\theta \colon R^\flat \to W(R^\flat)$$

is the Teichmüller lift coming from the surjection $\theta \colon W(R^\flat) \to R$.

Theorem 1.2 (cf. Theorem 6.4)**.** *The composition*[7]

$$T_p(R^\times) \to \pi_2(K(R;\mathbb{Z}_p)) \xrightarrow{\mathrm{ctr}} \pi_2(\mathrm{TC}(R;\mathbb{Z}_p)) \cong \widehat{\mathbb{\Delta}}_R^{\varphi=\tilde{\xi}}$$

is given by the negative of the q-logarithm

$$x \mapsto \log_q([x^{1/p}]_{\tilde{\theta}}) := \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(n-1)/2} \frac{([x^{1/p}]_{\tilde{\theta}} - 1)([x^{1/p}]_{\tilde{\theta}} - q) \ldots ([x^{1/p}]_{\tilde{\theta}} - q^{n-1})}{[n]_q}.$$

Here

$$[x^{1/p}]_{\tilde{\theta}} \colon R^\flat \to \widehat{\mathbb{\Delta}}_R$$

is the Teichmüller lift of $x^{1/p}$ coming from the surjection

$$\tilde{\theta} := \theta \circ \varphi^{-1} \colon W(R^\flat) \to R$$

and we embed

$$T_p(R^\times) \subseteq R^\flat,\ (r_0 \in R^\times[p], r_1, \ldots) \mapsto (1, r_0, r_1, \ldots).$$

By

$$[n]_q := \frac{q^n - 1}{q - 1}$$

we denote the the q-analog of $n \in \mathbb{Z}$.

Remark 1.3. A similar result can be found in [11, Lemma 4.2.3.], but only before p-completion and on π_1, which makes it impossible to deduce Theorem 1.2 from their result.

[7] Cf. Section 6 for a precise description of the isomorphism $\pi_2(\mathrm{TC}(R;\mathbb{Z}_p)) \cong \widehat{\mathbb{\Delta}}_R^{\varphi=\tilde{\xi}}$.

Remark 1.4. The appearance of the two different Teichmüller lifts can be avoided as

$$[x]_\theta = [x^{1/p}]_{\tilde{\theta}}$$

for $x \in R^\flat$. In the definition of q we took the classical one, but in the definition of $\log_q$ we used $[-]_{\tilde{\theta}}$ as it fits better with the perspective of [3].

As a consequence of [7] and Theorem 1.2, one gets the following result.

Corollary 1.5. *Let R be a quasi-regular semiperfectoid $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. The map*

$$\log_q([(-)^{1/p}]_{\tilde{\theta}}) : T_p(R^\times) \to \widehat{\mathbb{\Delta}}_R^{\varphi=\tilde{\xi}}$$

is a bijection.

This corollary is used in [1], which studies a prismatic version of Dieudonné theory for p-divisible groups, and was our original motivation for proving Theorem 1.2.

Here is a short description of the proof of Theorem 1.2. By testing the universal case $R = \mathbb{Z}^{\mathrm{cycl}}\langle x^{1/p^\infty}\rangle/(x-1)$ one is reduced to the case where $(p, \tilde{\xi})$ form a regular sequence on $\widehat{\mathbb{\Delta}}_R$, i.e., the prism $(\widehat{\mathbb{\Delta}}_R, \tilde{\xi})$ is *transversal* (cf. Definition 3.2). In this situation, we prove that the reduction map

$$\widehat{\mathbb{\Delta}}_R^{\varphi=\tilde{\xi}} \hookrightarrow \mathcal{N}^{\geq 1}\widehat{\mathbb{\Delta}}_R/\mathcal{N}^{\geq 2}\widehat{\mathbb{\Delta}}_R$$

is *injective* (cf. Corollary 3.10). Thus it suffices to identify the composition

$$T_p(R^\times) \xrightarrow{\mathrm{ctr}} \widehat{\mathbb{\Delta}}_R^{\varphi=\tilde{\xi}} \to \mathcal{N}^{\geq 1}\widehat{\mathbb{\Delta}}_R/\mathcal{N}^{\geq 2}\widehat{\mathbb{\Delta}}_R.$$

Using the results [3] the quotient $\mathcal{N}^{\geq 1}\widehat{\mathbb{\Delta}}_R/\mathcal{N}^{\geq 2}\widehat{\mathbb{\Delta}}_R$ identifies with the p-completed Hochschild homology $\pi_2(\mathrm{HH}(R;\mathbb{Z}_p))$ (cf. Section 5) and therefore the above composition identifies with the p-completed Dennis trace. A straightforward computation then identifies the p-completed Dennis trace (cf. Section 2), which allows us to conclude. We expect the results in Section 2 to be known, in some form, to the experts, but we did not find the results anywhere in the literature.

Let us end this introduction by a remark and a question. One could try to reverse the perspective from Corollary 1.5 and try to recover a (very) special case of the result of Clausen-Mathew-Morrow (cf. [7]) regarding the cyclotomic trace map using the concrete description furnished by Theorem 1.2. If R is of characteristic p, we may arrange that $q = 1$ and then the q-logarithm becomes the honest logarithm

$$\log([-]_\theta) \colon T_p(R^\times) \to A_{\mathrm{crys}}(R)^{\varphi=p}.$$

In [22], it is proven (using the exponential) that the map $\log([-])$ is an isomorphism, when R is the quotient of a perfect ring modulo a regular sequence. If R is the quotient of a perfectoid ring by a finite regular sequence and is p-torsion free, it is not difficult to deduce from Scholze-Weinstein's result that the map

$$\log_q([(-)^{1/p}]_{\tilde{\theta}}) \colon T_p(R^\times) \to \widehat{\mathbb{\Delta}}_R^{\varphi=\tilde{\xi}}$$

is a bijection when p is odd. Is there a way to prove it directly in general, for any p and any quasi-regular semiperfectoid ring ?

1.3. **Plan of the paper.** In Section 2 we concretely identify the p-completed Dennis trace on the Tate module of units (cf. Proposition 2.5) in the form we need it. In Section 3 we prove the crucial injectivity statement, namely Corollary 3.10, for transversal prisms. In Section 4 we make sense of the q-logarithm. Finally, in Section 6 we prove our main result Theorem 1.2 and its consequence, Corollary 1.5.

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2. The p-completed Dennis trace in degree 2

Fix some prime p and let A be a commutative ring. The aim of this section is to concretely describe the p-completed Dennis trace

$$T_p(A^\times) \to \pi_2(K(A;\mathbb{Z}_p)) \xrightarrow{\mathrm{Dtr}} \pi_2(\mathrm{HH}(A;\mathbb{Z}_p))$$

in degree 2 (cf. Corollary 2.2) and to deduce from this a description of the composite map

$$T_p(A^\times) \to \pi_2(\mathrm{HH}(A;\mathbb{Z}_p)) \to \pi_2(\mathrm{HH}(A/R;\mathbb{Z}_p))$$

when A is the quotient of a p-complete ring R by a finite regular sequence. Here

$$K(A;\mathbb{Z}_p)$$

denotes the p-completed (connective) K-theory spectrum of A and

$$\mathrm{HH}(A;\mathbb{Z}_p) \text{ resp. } \mathrm{HH}(A/R;\mathbb{Z}_p)$$

the p-completed Hochschild homology of A as a $\mathbb{Z}$-algebra resp. as an R-algebra (if A is an R-algebra for some commutative ring R), cf. [3, Section 2.2.].

Let us recall the construction of the first map $T_p(A^\times) \to \pi_2(K(A;\mathbb{Z}_p))$. Let

$$\mathrm{GL}(A) = \varinjlim_r \mathrm{GL}_r(A)$$

be the infinite general linear group over A. There is a canonical inclusion

$$A^\times = \mathrm{GL}_1(A) \to \mathrm{GL}(A)$$

of groups which on classifying spaces induces a map

$$B(A^\times) \to B(\mathrm{GL}(A)).$$

Composing with the morphism to Quillen's $+$-construction yields a canonical morphism

$$B(A^\times) \to B\mathrm{GL}(A) \to K(A) := B\mathrm{GL}(A)^+ \times K_0(A)$$

into the K-theory space

$$K(A) := B\mathrm{GL}(A)^+ \times K_0(A)$$

of A. After p-completion of spaces[8] we obtain a canonical morphism

$$\iota\colon B(A^\times)^\wedge_p \to K(A;\mathbb{Z}_p) := K(A)^\wedge_p.$$

[8]We use space as a synonym for Kan complex.

We recall (cf. [16, Theorem 10.3.2.]) that the space $B(A^\times)^\wedge_p$ has two non-trivial homotopy groups which are given by

$$\pi_1(B(A^\times)^\wedge_p) \cong H^0(R\varprojlim_n (A^\times \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n))$$

and

$$\pi_2(B(A^\times)^\wedge_p) \cong H^{-1}(R\varprojlim_n (A^\times \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n) \cong T_p(A^\times).$$

In degree 2 we thus get a morphism

$$T_p(A^\times) = \pi_2(B(A^\times)^\wedge_p) \to \pi_2(K(A;\mathbb{Z}_p)),$$

which is the first constituent of the map

$$T_p(A^\times) \to \pi_2(K(A;\mathbb{Z}_p)) \xrightarrow{\mathrm{Dtr}} \pi_2(\mathrm{HH}(A;\mathbb{Z}_p))$$

we want to describe. Now we turn to the construction of the Dennis trace

$$\mathrm{Dtr}\colon K(A) \to \mathrm{HH}(A).$$

as presented in [15, Section 8.4.].

By definition it will (on homotopy groups) factor through the integral group homology of $\mathrm{GL}(A)$, i.e., through $H_*(B\mathrm{GL}(A),\mathbb{Z})$, which is by definition (and the Dold-Kan correspondence) the homotopy of the space $\mathbb{Z}[B\mathrm{GL}(A)]$ obtained by taking the free simplicial abelian group on the simplicial $B\mathrm{GL}(A)$. As the $+$-construction

$$B\mathrm{GL}(A) \to B\mathrm{GL}(A)^+$$

is an equivalence on integral homology (cf. [26, Chapter IV.Theorem 1.5.]) the morphism

$$\mathbb{Z}[B\mathrm{GL}(A)] \simeq \mathbb{Z}[B\mathrm{GL}(A)^+]$$

is an equivalence of simplicial abelian groups and using the canonical inclusion

$$B\mathrm{GL}(A)^+ \to \mathbb{Z}[B\mathrm{GL}(A)^+]$$

we arrive at a canonical morphism

$$K(A) \to B\mathrm{GL}(A)^+ \to \mathbb{Z}[B\mathrm{GL}(A)^+] \simeq \mathbb{Z}[B\mathrm{GL}(A)].$$

We observe that for $r = 1$ the morphism $B\mathrm{GL}_1(A) \to B\mathrm{GL}_1(A)^+$ is an equivalence as $\mathrm{GL}_1(A) = A^\times$ is abelian. Thus there is a commutative diagram (up to homotopy)

$$\begin{tikzcd} B\mathrm{GL}_1(A) \arrow[r] \arrow[d] & \mathbb{Z}[B\mathrm{GL}_1(A)] \arrow[d] \\ K(A) \arrow[r] & \mathbb{Z}[B\mathrm{GL}(A)] \end{tikzcd}$$

with each morphism being the canonical one.

Let us explain the construction the Dennis trace map

$$\mathrm{Dtr}'\colon \mathbb{Z}[B\mathrm{GL}(A)] \to \mathrm{HH}(A/\mathbb{Z})$$

from integral group homology to Hochschild homology (following [15, Chapter 8.4.]). For this we need to recall first the definition of (derived) Hochschild homology (cf. [3, Section 2.2.]). The Dennis trace from K-theory will then be defined as the composition

$$\mathrm{Dtr}\colon K(A) \to \mathbb{Z}[B\mathrm{GL}(A)] \xrightarrow{\mathrm{Dtr}'} \mathrm{HH}(A/\mathbb{Z}).$$

If R is an arbitrary (commutative) ring and A an R-algebra (assumed to be commutative) the (derived) Hochschild homology $\mathrm{HH}(A/R)$ is defined as the geometric realization of the simplicial object

$$\mathrm{HH}(A/R) := \varinjlim_{\Delta^{\mathrm{op}}} A^{\otimes_R^{\mathbb{L}}{}^{n+1}}.$$

More concretely, if $B_\bullet$ is a commutative differential algebra resolving A, i.e.,

$$B_\bullet = \bigoplus_{n\geq 0} B_n$$

is a commutative differential R-algebra with a quasi-isomorphism $B_\bullet \cong A$ which is a morphism of algebras such that each B_n is flat over R, then the n-fold tensor product

$$B_\bullet \otimes_R \ldots \otimes_R B_\bullet$$

with differential extended multiplicatively is quasi-isomorphic to the n-fold derived tensor product $A \otimes_R^{\mathbb{L}} \otimes \ldots \otimes_R^{\mathbb{L}} A$ and the derived Hochschild homology $\mathrm{HH}(A/R)$ is computed by the totalization of the bicomplex

$$\begin{tikzcd}
\ldots \arrow[r] & (B_\bullet \otimes_R B_\bullet \otimes_R B_\bullet)_0 \arrow[r] & (B_\bullet \otimes_R B_\bullet)_0 \arrow[r] & B_0 \\
\ldots \arrow[r] & (B_\bullet \otimes_R B_\bullet \otimes_R B_\bullet)_1 \arrow[r] \arrow[u] & (B_\bullet \otimes_R B_\bullet)_1 \arrow[r] \arrow[u] & B_1 \arrow[u] \\
 & \ldots \arrow[r] \arrow[u] & \ldots \arrow[r] \arrow[u] & \ldots \arrow[u]
\end{tikzcd}$$

The first horizontal differentials are the (graded) maps

$$B_\bullet \otimes_R B_\bullet \to B_\bullet,\ a \otimes b \mapsto ab - (-1)^{|a||b|} ba$$

respectively

$$B_\bullet \otimes_R B_\bullet \otimes_R B_\bullet \to B_\bullet \otimes_R B_\bullet,\ a \otimes b \otimes c \mapsto ab \otimes c - a \otimes bc + (-1)^{|c||a|} ca \otimes b.$$

By convention we set

$$\mathrm{HH}(A) := \mathrm{HH}(A/\mathbb{Z}).$$

Moreover, we denote by

$$\mathrm{HH}(A/R; \mathbb{Z}_p)$$

the p-completion of the Hochschild homology $\mathrm{HH}(A/R)$. After choosing a resolution $B_\bullet \cong A$ as above the p-completed Hochschild homology can then be calculated, if R is p-torsion free, by taking the total complex of the bicomplex

$$\begin{tikzcd}
\ldots \arrow[r] & ((B \otimes_R B \otimes_R B)_0)_p^\wedge \arrow[r] & ((B \otimes_R B)_0)_p^\wedge \arrow[r] & (B_0)_p^\wedge \\
\ldots \arrow[r] & ((B \otimes_R B \otimes_R B)_1)_p^\wedge \arrow[r] \arrow[u] & ((B \otimes_R B)_1)_p^\wedge \arrow[r] \arrow[u] & (B_1)_p^\wedge \arrow[u] \\
 & \ldots \arrow[r] \arrow[u] & \ldots \arrow[r] \arrow[u] & \ldots \arrow[u]
\end{tikzcd}$$

where each term is p-adically completed (note that in the totalization only finite direct sums appear, which implies that the totalization remains p-adically complete). Clearly, the derived Hochschild homology is functorial in the pair (A, R). By construction the Dennis trace

$$\mathrm{Dtr}'\colon \mathbb{Z}[B\mathrm{GL}(A)] \to \mathrm{HH}(A)$$

will be given as the colimit of compatible maps[9]

$$\mathrm{Dtr}'_r\colon \mathbb{Z}[B\mathrm{GL}_r(A)] \to \mathrm{HH}(A).$$

We content ourselves to only describe the case $r = 1$ and A flat over $\mathbb{Z}$, the only case relevant for us. Then the map Dtr'_1 is the linear extension of a map

$$BA^\times \to \mathrm{HH}(A)$$

which in simplicial degree n is given by

$$(a_1, \dots, a_n) \mapsto \frac{1}{a_1 \dots a_n} \otimes a_1 \otimes \dots \otimes a_n.$$

We can conclude that the composition

$$\mathrm{Dtr}_1\colon BA^\times \to K(A) \to \mathbb{Z}[B\mathrm{GL}(A)] \to \mathrm{HH}(A)$$

we are interested in is given by the map

$$BA^\times \to \mathbb{Z}[BA^\times] \xrightarrow{\mathrm{Dtr}'_1} \mathrm{HH}(A)$$

described above. On π_1 this map is very familiar. We recall that there is a canonical[10] isomorphism

$$\pi_1(\mathrm{HH}(A/R)) \cong \Omega^1_{A/R},\ a \otimes b \mapsto adb$$

with inverse $adb \mapsto a \otimes b$. Thus on homotopy the map Dtr_1 is given by the dlog-map

$$A^\times \cong \pi_1(BA^\times) \to \pi_1(\mathrm{HH}(A)) \cong \Omega^1_{A/\mathbb{Z}},\ a \mapsto \mathrm{dlog}(a) := \frac{da}{a}.$$

In the following we want to describe, on π_2, the p-adic completion

$$\mathrm{Dtr}_1^\wedge\colon (BA^\times)^\wedge_p \to \mathrm{HH}(A; \mathbb{Z}_p)$$

of the Dennis trace. This turns out to be more tricky. Let us define

$$h\colon BA^\times \to \mathbb{Z}[BA^\times].$$

as the canonical inclusion (the "Hurewicz morphism"). The map

$$\mathrm{Dtr}'_1\colon \mathbb{Z}[BA^\times] \to \mathrm{HH}(A)$$

is $\mathbb{Z}$-linear and thus its effect after p-completion is easy to calculate if A is p-torsion free : simply apply the p-adic completion of abelian groups in each simplicial degree. Thus our next task is to describe the p-adic completion of the Hurewicz map

$$h\colon BA^\times \to \mathbb{Z}[BA^\times]$$

on π_2. More generally, we will do this for an arbitrary abelian group G.

[9]Here compatible means up to some homotopy. To obtain strict compatibility one has to use the normalised Hochschild complex, cf. [15, Section 8.4.]

[10]Up to the choice of a tensor factor.

Proposition 2.1. *Let G be an abelian group then the map*

$$h_p^\wedge \colon T_p(G) \cong \pi_2((BG)_p^\wedge) \to \pi_2(\mathbb{Z}[BG]_p^\wedge)$$

induced by the p-completed Hurewicz sends an element

$$(g_1, g_2, \ldots) \in T_p(G) = \varprojlim_n G[p^n]$$

to the class represented by

$$\sum_{n=0}^{\infty} p^{n-1}((1, g_n) + (g_n, g_n) + \ldots + (g_n^{p-1}, g_n)),$$

where we set $g_0 := 1$.

Proof. By definition

$$T_p(G) = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G)$$

an thus by naturality it suffices to determine the image in $\pi_2(\mathbb{Z}[B\mathbb{Q}_p/\mathbb{Z}_p]_p^\wedge)$ under $\pi_2(h_p^\wedge)$ of the canonical generator

$$(\frac{1}{p} + \mathbb{Z}_p, \frac{1}{p^2} + \mathbb{Z}_p, \ldots) \in T_p(\mathbb{Q}_p/\mathbb{Z}_p).$$

To avoid confusion let us for g in $G = \mathbb{Z}_p$, $\mathbb{Q}_p$ or $\mathbb{Q}_p/\mathbb{Z}_p$ denote the corresponding element in $\mathbb{Z}[G]$ by t^g.

To calculate the image of this canonical generator we will use the canonical short exact sequence

$$0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

with associated fiber sequence

$$B\mathbb{Z}_p \to B\mathbb{Q}_p \to B\mathbb{Q}_p/\mathbb{Z}_p$$

on classifying spaces. Moreover, there is a commutative diagram

$$\begin{tikzcd}
B\mathbb{Z}_p \arrow[r] \arrow[d] & B\mathbb{Q}_p \arrow[r] \arrow[d] & B\mathbb{Q}_p/\mathbb{Z}_p \arrow[d] \\
\mathbb{Z}[B\mathbb{Z}_p] \arrow[r] & \mathbb{Z}[B\mathbb{Q}_p] \arrow[r] & \mathbb{Z}[B\mathbb{Q}_p/\mathbb{Z}_p]
\end{tikzcd}$$

where the lower sequence computes integral homology, but is not a fiber sequence of spaces anymore. Let K be the homotopy colimit in the diagram

$$\begin{tikzcd}
\mathbb{Z}[B\mathbb{Z}_p] \arrow[r, "\pi"] \arrow[d, "\iota"] & \mathbb{Z}[B\{1\}] \arrow[d] \\
\mathbb{Z}[B\mathbb{Q}_p] \arrow[r] & K
\end{tikzcd}$$

where $\{1\}$ denotes the trivial group. In other words, K is the complex with n-th term (in homological notation) given by

$$K_n = \mathbb{Z}[\{1\}] \oplus \mathbb{Z}[\mathbb{Q}_p^n] \oplus \mathbb{Z}[\mathbb{Z}_p^{n-1}]$$

and differential

$$(a, b, c) \mapsto (da - \pi(c), db + \iota(c), -dc)$$

(here we freely used the Dold-Kan correspondence to pass from simplicial abelian groups to chain complexes). As $\mathbb{Z}_p$ maps to 0 in $\mathbb{Q}_p/\mathbb{Z}_p$ there is the natural morphism

$$K \to \mathbb{Z}[B\mathbb{Q}_p/\mathbb{Z}_p],\ (a,b,c) \to a + \overline{b}$$

(using the inclusion $\{1\}$ and the projection $B\mathbb{Q}_p \to B\mathbb{Q}_p/\mathbb{Z}_p,\ b \mapsto \overline{b}$). Now, we pass to p-adic completion. The p-completion of $B\mathbb{Q}_p$ is weakly contractible because the derived p-adic completion of $\mathbb{Q}_p$ vanishes and $B\mathbb{Q}_p^\wedge$ has homotopy groups given by the cohomology of

$$R\varprojlim_n (\mathbb{Q}_p \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n) = 0$$

(cf. [16, Theorem 10.3.2.]). After p-adic completion the sequence

$$B{\mathbb{Z}_p}^\wedge \to B{\mathbb{Q}_p}^\wedge \simeq * \to B\mathbb{Q}_p/{\mathbb{Z}_p}^\wedge$$

remains a fiber sequence[11], thus the connecting homomorphism yields an isomorphism

$$T_p(\mathbb{Q}_p/\mathbb{Z}_p) = \pi_2(B\mathbb{Q}_p/{\mathbb{Z}_p}^\wedge) \cong \pi_1(B{\mathbb{Z}_p}^\wedge) \cong \mathbb{Z}_p$$

sending the canonical generator $\gamma := (\frac{1}{p} + \mathbb{Z}_p, \frac{1}{p^2} + \mathbb{Z}_p, \ldots)$ to $1 \in \mathbb{Z}_p$ (as can be calculated using the natural isomorphism of the homotopy of the p-completion with the cohomology of the derived p-completion). Moreover, there is a commutative diagram

$$\begin{tikzcd}
\pi_2((B\mathbb{Q}_p/\mathbb{Z}_p)^\wedge) \arrow[rr] \arrow[d] & & \pi_1((B\mathbb{Z}_p)^\wedge) \arrow[d] \\
\pi_2(\mathbb{Z}[B\mathbb{Q}_p/\mathbb{Z}_p]^\wedge) & \pi_2(K_p^\wedge) \arrow[l] \arrow[r, "\cong"] & \pi_1(\mathbb{Z}[B\mathbb{Z}_p]^\wedge)
\end{tikzcd}$$

and we can use it to compute the image of $(1/p+\mathbb{Z}_p, 1/p^2+\mathbb{Z}_p, \ldots)$ in $\pi_2(\mathbb{Z}[B\mathbb{Q}_p/\mathbb{Z}_p]^\wedge)$. The canonical generator $1 \in \pi_1((B\mathbb{Z}_p)^\wedge)$ maps to the element $t = t^1 \in \mathbb{Z}[\mathbb{Z}_p]$ in degree 1 of $\mathbb{Z}[B\mathbb{Z}_p]$. In order to lift it to

$$\pi_2(K_p^\wedge)$$

we have to find a cycle $(a,b,c) \in (K_p^\wedge)_2 = \mathbb{Z}_p \oplus \mathbb{Z}[\mathbb{Q}_p^2]^\wedge \oplus \mathbb{Z}[\mathbb{Z}_p]$ such that $c = t$. By definition

$$d(a,b,t) = (da - 1, db + t, 0)$$

and thus we can set $a = 1 \in \mathbb{Z}_p = \mathbb{Z}[B\{1\}]_2^\wedge$ (where the subscript indicates the simplicial degree). The element

$$b := \sum_{n=1}^{\infty} p^{n-1}((1, t^{1/p^n}) + (t^{1/p^n}, t^{1/p^n}) + \ldots + (t^{p-1/p^n}, t^{1/p^n})) \in \mathbb{Z}[\mathbb{Q}_p^2]$$

satisfies $db = -t$. Namely, one calculates

$$\begin{aligned} t &= t^{1/p} + t^{p-1/p} - d(t^{p-1/p}, t^{1/p}) \\ &= \ldots = pt^{1/p} - d((t^{p-1/p}, t^{1/p}) + \ldots + (t^{1/p}, t^{1/p}) + (1, t^{1/p})) \end{aligned}$$

because

$$d(g,h) = g + h - gh$$

for $(g,h) \in \mathbb{Z}[\mathbb{Q}_p^2]^\wedge$. Then one iterates the formula, i.e., takes it with $t^{1/p}$ and t^{1/p^2} instead of t and $t^{1/p}$. As $p^n t^{1/p^n}$ tends to zero in $(\mathbb{Z}[\mathbb{Q}_p])_p^\wedge$ we arrive at the formula

[11] As this can be checked on homotopy groups.

$-t = db$. Thus the element $(1, b, t) \in \mathbb{Z}[K_p^\wedge]$ is a cycle as we searched for. Its image in $\pi_2(\mathbb{Z}[B\mathbb{Q}_p/\mathbb{Z}_p]_p^\wedge)$ is the class represented by the element

$$(1, 1) + \overline{b}$$

as claimed. □

We arrive at our first description of the p-completed Dennis trace.

Corollary 2.2. *Under the p-completed Dennis trace $(BA^\times)_p^\wedge \to \mathrm{HH}(A; \mathbb{Z}_p)$ the image of an element*

$$(a_1, a_2, \ldots) \in T_p(A^\times) = \pi_2((BA^\times)_p^\wedge)$$

is represented by the cycle

$$\sum_{n=0}^{\infty} p^{n-1}(a_n^{-1} \otimes 1 \otimes a_n + \ldots + a_n^{-p} \otimes a_n^{p-1} \otimes a_n)$$

with setting $a_0 = 1$.

Proof. This follows from Proposition 2.1 and the definition of the Dennis trace

$$\mathrm{Dtr}' \colon \mathbb{Z}[B\mathrm{GL}_1(A)] \to \mathrm{HH}(A)$$

from integral group homology to Hochschild homology. □

Next we want to describe the Dennis trace in the context that A is p-complete and the quotient $A = R/I$ with I generated by some regular sequence for some arbitrary (p, I)-complete ring R.

We want to describe the image of some element $(a_1, a_2, \ldots) \in T_p(A^\times)$ under the composition

$$T_p(A^\times) \xrightarrow{\mathrm{Dtr}} \pi_2(\mathrm{HH}(A; \mathbb{Z}_p)) \to \pi_2(\mathrm{HH}(A/R; \mathbb{Z}_p)) \cong I/I^2,$$

where in the last isomorphism we used the following simplification.

Lemma 2.3. *There is a natural isomorphism*

$$I/I^2 \cong \pi_2(\mathrm{HH}(A/R)).$$

and the Hochschild homology $\mathrm{HH}(A/R)$ is already p-adically complete.

Proof. The p-completeness follows as the Koszul complex for a generating regular sequence of I is a flat resolution of A over R and the associated bicomplex has p-adically complete terms as R is p-adically complete. The first assertion follows from the HKR-filtration on $\mathrm{HH}(A/R; \mathbb{Z}_p)$ from [3, Section 2.2.] and the fact there is a canonical isomorphism

$$I/I^2 \cong L_{A/R}[-1]$$

of I is generated by a regular sequence. □

In the case that $I = (f)$ is generated by some regular element f in R we make the isomorphism explicit. The Koszul complex $B_\bullet = (I \to R)$ is a flat resolution of

A and the bicomplex computing $\mathrm{HH}(A/R)$ reads as

$$\begin{tikzcd}
\dots \arrow[r] & R \arrow[r, "1"] & R \arrow[r, "0"] & R \\
\dots \arrow[r] & I \oplus I \oplus I \arrow[u] \arrow[r, "d_h"] & I \oplus I \arrow[u] \arrow[r, "0"] & I \arrow[u] \\
 & \dots \arrow[u] \arrow[r] & I \otimes_R I \arrow[u, "d_v"] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

with

$$d_h \colon I \oplus I \oplus I \to I \oplus I,\ (a,b,c) \mapsto (a+b+c, 0)$$

and

$$d_v \colon I \otimes_R I \to I \oplus I,\ a \otimes b \mapsto (ab, -ba)$$

while the other maps are the canonical summation maps. The isomorphism

$$I/I^2 \cong \pi_2(\mathrm{HH}(A/R))$$

is now induced by the map

$$I \to I \oplus I,\ a \mapsto (0, a)$$

into the $(1,1)$-component of the double complex.

We turn to the description of the map

$$T_p(A^\times) \to I/I^2$$

induced by the Dennis trace map (in the case that I is generated by some regular sequence). We recall the following standard lemma.

Lemma 2.4. *Let R be a ring, $I \subseteq R$ and ideal and assume that R is (p, I)-adically complete. Then the canonical map*

$$\varprojlim_{x \mapsto x^p} R \to \varprojlim_{x \mapsto x^p} A$$

with $A = R/I$ is bijective.

Proof. It suffices to construct a well-defined, multiplicative map

$$[-] \colon \varprojlim_{x \mapsto x^p} R/I \to R$$

reducing to the first projection modulo I. Let

$$r := (r_0, r_1, \dots) \in \varprojlim_{x \mapsto x^p} A$$

be a p-power compatible system of elements in R/I with lifts $r_i' \in R$ of each r_i. Then the limit

$$\lim_{n \to \infty} (r_n')^{p^n}$$

exists and is independent of the lift. Thus

$$[r] := \lim_{n \to \infty} (r_n')^{p^n}$$

defines the desired map. $\square$

The morphism

$$[-]\colon \varprojlim_{x\mapsto x^p} R/I \to R$$

is the Teichmüller lift for the surjection $\pi\colon R \to R/I$. If we want to make its dependance of the surjection clear, we write $[-]_\pi$.

For some element $\gamma = (a_1, a_2, \ldots) \in T_p(A^\times)$ we obtain by the previous lemma a p-power compatible system $(r_0, r_1, \ldots) \in \varprojlim_{x\mapsto x^p} R$ of units in R reducing to the sequence $(1, a_1, a_2, \ldots)$ in A (note the additional 1 at the sequence). We define

$$[\gamma] := r_0.$$

Clearly, $[\gamma] \in 1 + I$. Finally, we arrive at the following concrete description of the Dennis trace map.

Proposition 2.5. *Let R be a p-complete ring, $I \subseteq R$ an ideal generated by a regular sequence such that R is (p, I)-adically complete. Let $A = R/I$. Then the composition*

$$T_p(A^\times) \cong \pi_2((BA^\times)^\wedge_p) \xrightarrow{\mathrm{Dtr}} \pi_2(\mathrm{HH}(A/R)^\wedge_p) \cong I/I^2$$

is given by sending $\gamma \in T_p(A^\times)$ to $\frac{1-[\gamma]}{[\gamma]} = [\gamma^{-1}] - 1$.

Proof. By naturality we may assume

$$R = \mathbb{Z}[\mathbb{Q}_p]^\wedge_p,$$

$$I = (t-1),$$

$$A = R/I \cong \mathbb{Z}[\mathbb{Q}_p/\mathbb{Z}_p]^\wedge_p$$

and

$$\gamma = (t^{1/p+\mathbb{Z}_p}, t^{1/p^2+\mathbb{Z}_p}, \ldots) \in T_p(\mathbb{Z}[A^\times]^\wedge_p).$$

Then $[\gamma] = t$. We let $B_\bullet$ be the Koszul resolution of A as an R-algebra, i.e., $B_\bullet$ is given by the complex

$$B_\bullet := (\ldots \to 0 \to I \to R).$$

We can calculate the p-completed Hochschild homology $\mathrm{HH}(A/\mathbb{Z}; \mathbb{Z}_p)$ either using the chain complex

$$C_\bullet := (\ldots \to A \otimes^\wedge_{\mathbb{Z}_p} A \otimes^\wedge_{\mathbb{Z}_p} A \to A \otimes^\wedge_{\mathbb{Z}_p} A \to A)$$

or the total complex

$$D_\bullet := \mathrm{Tot}(\ldots \to B_\bullet \otimes^\wedge_{\mathbb{Z}_p} B_\bullet \otimes^\wedge_{\mathbb{Z}_p} B_\bullet \to B_\bullet \otimes^\wedge_{\mathbb{Z}_p} B_\bullet \to B_\bullet).$$

Concretely, let us look at the element

$$y := \sum_{n=0}^{\infty} p^{n-1}(t^{-1/p^n+\mathbb{Z}_p} \otimes 1 \otimes t^{1/p^n+\mathbb{Z}_p} + \ldots + t^{-p/p^n+\mathbb{Z}_p} \otimes t^{p-1/p^n+\mathbb{Z}_p} \otimes t^{1/p^n+\mathbb{Z}_p})$$

in C_2. It defines a cycle in $C_\bullet$ and by Corollary 2.2 it represents the image of γ in $\pi_2(\mathrm{HH}(A; \mathbb{Z}_p))$. We want to lift it to a cycle in D_2. First, we lift y to

$$y' := \sum_{n=0}^{\infty} p^{n-1}(t^{-1/p^n} \otimes 1 \otimes t^{1/p^n} + \ldots + t^{-p/p^n} \otimes t^{p-1/p^n} \otimes t^{1/p^n})$$

which however is not a cycle in $D_\bullet$. Let us compute dy'. There is a commutative diagram

$$\begin{tikzcd} \mathbb{Z}[\mathbb{Q}_p \times \mathbb{Q}_p]_p^\wedge \arrow[r, "\alpha"] \arrow[d, "d"] & \mathbb{Z}[\mathbb{Q}_p \times \mathbb{Q}_p \times \mathbb{Q}_p]_p^\wedge \cong (R \otimes_{\mathbb{Z}} R \otimes_{\mathbb{Z}} R)_p^\wedge \arrow[d, "d"] \\ \mathbb{Z}[\mathbb{Q}_p]_p^\wedge \arrow[r, "\beta"] & \mathbb{Z}[\mathbb{Q}_p \times \mathbb{Q}_p]_p^\wedge \cong (R \otimes_{\mathbb{Z}} R)_p^\wedge \end{tikzcd}$$

where the horizontal arrows are the ones defining the Dennis trace

$$\mathbb{Z}[B\mathbb{Q}_p]_p^\wedge \to \mathrm{HH}(R/\mathbb{Z}; \mathbb{Z}_p),$$

i.e., are given by $(g,h) \to 1/gh \otimes g \otimes h$ resp. $g \to 1/g \otimes g$. By construction the element $y' = \alpha(x')$ is the image of

$$x' = \sum_{n=0}^{\infty} p^{n-1}((1, t^{1/p^n}) + (t^{1/p^n}, t^{1/p^n}) + \ldots + (t^{p-1/p^n}, t^{1/p^n})) \in \mathbb{Z}[\mathbb{Q}_p \times \mathbb{Q}_p]_p^\wedge.$$

In the proof of Proposition 2.1 we calculated $dx' = 1 - t$ (note that $d((1,1)) = 1$ under the differential

$$\mathbb{Z}[\mathbb{Q}_p \times \mathbb{Q}_p] \to \mathbb{Z}[\mathbb{Q}_p],\ (g,h) \mapsto g + h - gh).$$

Thus, we can conclude

$$dy' = \beta(1-t) = 1 \otimes 1 - \frac{1}{t} \otimes t.$$

In order to lift y to a cycle in D_2 we have to find some element

$$z \in (R \otimes_{\mathbb{Z}} I \oplus I \otimes_{\mathbb{Z}} R)_p^\wedge$$

such that $dz' = dy'$ (i.e., a preimage of dy' under the horizontal differential in the double complex) as then $y' - z'$ will be a cycle in D_2 lifting the cycle $y \in C_2$. We can write

$$dy' = 1 \otimes 1 - \frac{1}{t} \otimes t = 1 \otimes 1 - \frac{1}{t} \otimes 1 + \frac{1}{t} \otimes 1 - \frac{1}{t} \otimes t = 1 \otimes \frac{t-1}{t} + \frac{1}{t} \otimes (1-t)$$

and thus

$$dy' = dz$$

with

$$z := (1 \otimes \frac{t-1}{t}, \frac{1}{t} \otimes (1-t)).$$

To summarize, the cycle $y - z \in D_2$ represents the image of $\gamma \in T_p(A^\times)$ in $\pi_2(\mathrm{HH}(A; \mathbb{Z}_p))$. Mapping further to $\mathrm{HH}(A/R; \mathbb{Z}_p) = \mathrm{HH}(A/R)$ means to replace the tensor products over $\mathbb{Z}$ in $D_\bullet$ by tensor products over R. Using Lemma 2.3, or better the concrete example following it, we get that the image of γ in $\pi_2(\mathrm{HH}(A/R))$ is represented by the element $(\frac{t-1}{t}, \frac{1-t}{t})$ in $\pi_2(\mathrm{HH}(A/R))$, which maps to the element $\frac{1-t}{t}$ under the isomorphism $\pi_2(\mathrm{HH}(A/R)) \cong I/I^2$ from Lemma 2.3. This finishes the proof. □

We recall the following lemma. For a perfect ring S we denote its ring of Witt vectors by $W(S)$.

Lemma 2.6. *Let S be a perfect ring and let A be an $W(S)$-algebra. Then the canonical morphism*

$$\mathrm{HH}(A;\mathbb{Z}_p) \to \mathrm{HH}(A/W(S);\mathbb{Z}_p)$$

is an equivalence.

Proof. By the HKR-filtration from [3, Section 2.2] it suffices to see that the canonical morphism

$$L_{A/\mathbb{Z}} \to L_{A/W(S)}$$

of cotangent complexes is a p-adic equivalence, i.e., an equivalence after $-\otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p$. Computing the right hand side by polynomial algebras over $W(S)$ we see that it suffices to consider the case that A is p-torsion free. Then by base change

$$L_{A/\mathbb{Z}} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p \cong L_{(A/p)/\mathbb{F}_p}$$

resp.

$$L_{A/W(S)} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p \cong L_{(A/p)/S}$$

and the claim follows from the transitivity triangle

$$A/p \otimes^{\mathbb{L}}_{S} L_{S/\mathbb{F}_p} \to L_{(A/p)/\mathbb{F}_p} \to L_{(A/p)/S}$$

using that S is perfect which implies that the cotangent complex $L_{S/\mathbb{F}_p}$ of S over $\mathbb{F}_p$ vanishes. □

3. Transversal prisms

In this section we want to prove the crucial injectivity statement (Corollary 3.10) mentioned in the introduction. Let us recall the following definition from [4].

Definition 3.1. A *δ-ring* is a pair (A,δ), where A is a commutative ring, $\delta\colon A \to A$ a map of sets, with $\delta(0)=0$, $\delta(1)=0$, and

$$\delta(x+y) = \delta(x)+\delta(y)+\frac{x^p+y^p-(x+y)^p}{p}\ ;\ \delta(xy) = x^p\delta(y)+y^p\delta(x)+p\delta(x)\delta(y),$$

for all $x,y \in A$.

A *prism* (A,I) is a δ-ring A with an ideal I defining a Cartier divisor on $\mathrm{Spec}(A)$, such that A is derived (p,I)-adically complete and $p \in (I,\varphi(I))$.

Here, the map

$$\varphi\colon A \to A,\ x \mapsto \varphi(x) := x^p + p\delta(x)$$

denotes the lift of Frobenius induced from δ-structure on A. We will make the (usually harmless) assumption that $I = (\tilde{\xi})$ is generated by some distinguished element $\tilde{\xi} \in A$, i.e., $\tilde{\xi}$ is a non-zero divisor and $\delta(\tilde{\xi})$ is a unit.

Definition 3.2. We call a prism *transversal* if $(p,\tilde{\xi})$ is a regular sequence on A.

Let us fix a transversal prism (A,I). In particular, A is p-torsion free. Moreover, A is classically (p,I)-adically complete. Indeed, $(p,\tilde{\xi})$ being a regular sequence implies that

$$A \otimes^{\mathbb{L}}_{\mathbb{Z}[x,y]} \mathbb{Z}[x,y]/(x^n,y^n) \cong A/(p^n,\tilde{\xi}^n)$$

and therefore

$$A \cong R\varprojlim_n (A \otimes^{\mathbb{L}}_{\mathbb{Z}[x,y]} \mathbb{Z}[x,y]/(x^n,y^n) \cong R\varprojlim_n (A/(p^n,\tilde{\xi}^n)) \cong \varprojlim_n A/(p^n,\tilde{\xi}^n)$$

using Mittag-Leffler for the last isomorphism.

We set

$$I_r := I\varphi(I)\dots\varphi^{r-1}(I)$$

for $r \geq 1$ (where $\varphi^0(I) := I$). Then $I_r = (\tilde{\xi}_r)$ with

$$\tilde{\xi}_r := \tilde{\xi}\varphi(\tilde{\xi})\cdots\varphi^{r-1}(\tilde{\xi}).$$

Lemma 3.3. *For all $r \geq 1$ the element*

$$\varphi^r(\tilde{\xi})$$

is a non-zero divisor and $(\varphi^r(\tilde{\xi}), p)$ is again a regular sequence. In particular, the elements $\tilde{\xi}_r$, $r \geq 1$, are non-zero divisors.

Proof. Replacing $\tilde{\xi}$ by $\varphi(\tilde{\xi})$, which is again distinguished, it suffices to prove the statement for $r = 1$. Let $x \in A$ and assume $\varphi(\tilde{\xi})x = 0$. Then we get

$$0 = \varphi(\tilde{\xi}) = \tilde{\xi}^p x + p\delta(\tilde{\xi})x.$$

As $\delta(\tilde{\xi}) \in A^\times$ is a unit we see by reducing mod $\tilde{\xi}$ that $\tilde{\xi}$ divides x, i.e., there exists $y \in A$ with $\tilde{\xi}y = x$. As A is $\tilde{\xi}$-torsion free we get $\varphi(\tilde{\xi})y = 0$ and thus

$$x \in \bigcap_{i\geq 0} \tilde{\xi}^i A = 0$$

using $\tilde{\xi}$-adic completeness of A. As $p \in A$ is by assumption a non-zero divisor the sequence $(\varphi(\tilde{\xi}), p)$ is regular if and only if the sequence $(p, \varphi(\tilde{\xi}))$ is regular (cf. [23, Tag 07DW]). But $\varphi(\tilde{\xi})$ and $\tilde{\xi}^p$ agree modulo p, which shows that $(p, \varphi(\tilde{\xi}))$ is a regular sequence. □

Lemma 3.4. *The ring A is complete for the topology induced by the ideals I_r, i.e.,*

$$A \cong \varprojlim_r A/I_r.$$

Proof. First note that the ideal (p, I) is stable by φ. Namely, as $(p) \subseteq (p, I)$ this may be checked modulo (p), but for the characteristic p ring $A/(p)$ every ideal $J \subseteq A/I$ is stable under Frobenius. Thus we see $\varphi^r(I) \subseteq (p, I)$ for all $r \geq 1$ and thus

$$I_r \subseteq (p, I)^r.$$

In particular, $\bigcap_{r\geq 1} I_r = 0$ as A is separated for the (p, I)-adic topology. Now let $x_r \in I_r$ be elements. We have to show that the sequence

$$a_n := \sum_{r=1}^{n} x_r$$

converges in A. As $I_r \subseteq (p, I)^r$ the limit

$$a := \lim_{n\to\infty} a_n$$

for the (p, I)-adic topology exists. We want to see that it is also the limit for the topology given by the ideals I_r. Fix $n \geq 0$. It suffices to show that

$$a - a_n \in I_n$$

because then

$$a - a_m = a - a_n + a_n - a_m \in I_n$$

for all $m \geq n$ as $a_n - a_m = -\sum_{r=n+1}^{m} x_r \in I_n$. We calculate

$$a - a_n = \sum_{r=n+1}^{\infty} x_r = \sum_{r=n+1}^{\infty} \tilde{\xi}_n y_r$$

for some elements $y_r \in A$, which may be chosen to lie in $(p, I)^{r-n}$. As A is (p, I)-adically complete the sum

$$y := \sum_{r=n+1}^{\infty} y_r$$

converges thus in A. Hence

$$a - a_n = \tilde{\xi}_n y$$

lies in I_n. This finishes the proof. □

Lemma 3.5. *For $r \geq 1$ there is a congruence*

$$\varphi^r(\tilde{\xi}) \equiv pu \text{ modulo } (\tilde{\xi})$$

with $u \in A^\times$ some unit.

Proof. For $r = 1$ this follows from

$$\varphi(\tilde{\xi}) = \tilde{\xi}^p + p\delta(\xi)$$

because by definition of distinguishedness the element $\delta(\xi) \in A^\times$ is a unit. For $r \geq 2$ we compute

$$\varphi^r(\tilde{\xi}) = \varphi^{r-1}(\tilde{\xi}^p + p\delta(\tilde{\xi})) = \varphi^{r-1}(\tilde{\xi})^p + p\varphi^{r-1}(\delta(\tilde{\xi})).$$

By induction we may write $\varphi^{r-1}(\tilde{\xi}) = pu + a\tilde{\xi}$ with $u \in A^\times$ some unit and thus modulo $\tilde{\xi}$ we calculate

$$\varphi^r(\tilde{\xi}) \equiv (pu)^p + p\varphi(\delta(\tilde{\xi})) = p(\varphi(\delta(\tilde{\xi})) + p^{p-1}u^p)$$

with $\varphi(\delta(\tilde{\xi})) + p^{p-1}u^p \in A^\times$ some unit. □

Lemma 3.6. *For all $r \geq 1$ the sequences $(\varphi^r(\tilde{\xi}), \tilde{\xi})$ and $(\tilde{\xi}, \varphi^r(\tilde{\xi}))$ are again regular. Moreover, $I_r = \bigcap_{i=0}^{r-1} \varphi^i(I)$ for all $r \geq 1$.*

Proof. We can write $\varphi(\tilde{\xi}) = p\delta(\tilde{\xi}) + \tilde{\xi}^p$, where $\delta(\tilde{\xi}) \in A^\times$ is a unit. By Lemma 3.5 we get $\varphi^r(\tilde{\xi}) \equiv pu$ modulo $(\tilde{\xi})$ with $u \in A^\times$ a unit. As $(\tilde{\xi}, p)$ is a regular sequence we conclude (using [23, Tag 07DW] and Lemma 3.3) that $(\varphi^r(\tilde{\xi}), \tilde{\xi})$ is a regular sequence. To prove the last statement we proceed by induction on r. First note the following general observation: If R is some ring and (f, g) a regular sequence in R, then $(f) \cap (g) = (fg)$. In fact, if $r = sg \in (f) \cap (g)$, then $sg \equiv 0$ modulo f, hence $s \equiv 0$ modulo f as desired. Thus, it suffices to prove that $(\tilde{\xi}_r, \varphi^r(\tilde{\xi})$ is a regular sequence for $r \geq 1$ (recall that $\tilde{\xi}_r = \tilde{\xi}\varphi(\tilde{\xi}) \cdots \varphi^{r-1}(\tilde{\xi})$). By induction the morphism

$$A/(\tilde{\xi}_r) \to \prod_{i=0}^{r-1} A/(\varphi^i(\tilde{\xi}))$$

is injective. Hence, it suffices to show that for each $i = 0, \ldots, r-1$ the element $\varphi^r(\tilde{\xi})$ maps to a non-zero divisor in $A/(\varphi^i(\tilde{\xi}))$. But this follows from Lemma 3.5 which implies $\varphi^r(\tilde{\xi}) \equiv pu$ modulo $\varphi^i(\tilde{\xi})$ for some unit $u \in A^\times$. □

We can draw the following corollary.

Lemma 3.7. *Define* $\rho\colon A \to \prod_{r\geq 0} A/\varphi^r(I),\ x \mapsto (x \bmod \varphi^r(I))$. *Then* ρ *is injective.*

Proof. This follows from Lemma 3.4 and Lemma 3.6 as the kernel of ρ is given by $\bigcap_{r=1}^{\infty} \varphi^r(I) = \bigcap_{r=1}^{\infty} I_r = 0$. □

We now define the Nygaard filtration of the prism (A, I) (cf. [4, Definition 11.1]).

Definition 3.8. Define

$$\mathcal{N}^{\geq n}A := \{x \in A \mid \varphi^n(x) \in I^n A\},$$

the n-th filtration step of the Nygaard filtration.

By definition the Frobenius on A induces a morphism

$$\varphi\colon \mathcal{N}^{\geq 1}A \to I.$$

Note that we do not divide the Frobenius by $\tilde{\xi}$. Moreover, we define

$$\sigma\colon \prod_{i\geq 0} A/\varphi^i(I) \to \prod_{i\geq 0} A/\varphi^i(I),\ (x_0, x_1, \ldots) \mapsto (0, \varphi(x_0), \varphi(x_1), \ldots).$$

Here we use that if $a \equiv b \bmod \varphi^i(I)$, then $\varphi(a) \equiv \varphi(b) \bmod \varphi^{i+1}(I)$ to get that σ is well-defined. Then the diagram

$$\begin{tikzcd} \mathcal{N}^{\geq 1}A \arrow[r, "\rho"] \arrow[d, "\varphi"] & \prod_{i\geq 0} A/\varphi^i(I) \arrow[d, "\sigma"] \\ I \arrow[r, "\rho"] & \prod_{i\geq 0} A/\varphi^i(I) \end{tikzcd} \tag{1}$$

commutes where ρ is the homomorphism from Lemma 3.7.

Lemma 3.9. *The reduction map*

$$A^{\varphi=\tilde{\xi}} \to A/I,\ x \mapsto x \bmod (\tilde{\xi})$$

is injective.

Proof. Let $x \in A^{\varphi=\tilde{\xi}} \cap I$. We want to prove that $x = 0$. Clearly, $x \in \mathcal{N}^{\geq 1}A$. By Lemma 3.7 it suffices to prove that

$$x \equiv 0 \bmod \varphi^i(I)$$

for all $i \geq 0$. Write

$$\rho(x) = (x_0, x_1, \ldots)$$

By the commutativity of the square (Equation (1)) we get

$$\rho(\varphi(x)) = \sigma(\rho(x)) = (0, \varphi(x_0), \varphi(x_1), \ldots).$$

As $\varphi(x) = \tilde{\xi}x$ and therefore $\rho(\varphi(x)) = \tilde{\xi}\rho(x)$ we thus get

$$(\tilde{\xi}x_0, \tilde{\xi}x_1, \ldots) = (0, \varphi(x_0), \varphi(x_1), \ldots).$$

We assumed that $x \in I$, thus $x_0 = 0 \in A/I$. Now we use that $\tilde{\xi}$ is a non-zero divisor modulo $\varphi^i(I)$ (cf. Lemma 3.6) for $i > 0$. Hence, if $x_i = 0$, then

$$0 = \varphi(x_i) = \tilde{\xi}x_{i+1} \in A/\varphi^{i+1}(I)$$

implies $x_{i+1} = 0$. Beginning with $x_0 = 0$ this shows that $x_i = 0$ for all $i \geq 0$, which implies our claim. □

The same proof shows that also the reduction map

$$A^{\varphi=\tilde{\xi}^n} \to A/I$$

is injective for $n \geq 1$.

The following corollary is crucially used in Theorem 6.4.

Corollary 3.10. *The reduction map*

$$A^{\varphi=\tilde{\xi}} \to \mathcal{N}^{\geq 1}A/\mathcal{N}^{\geq 2}A$$

is injective.

Proof. Let $x \in A^{\varphi=\tilde{\xi}} \cap \mathcal{N}^{\geq 2}A$. Then

$$\tilde{\xi}x = \varphi(x) = \tilde{\xi}^2 y$$

for some $y \in A$. As $\tilde{\xi}$ is a non-zero divisor in A we get $x \in I = (\tilde{\xi})$. But then $x = 0$ by Lemma 3.9. □

4. The q-logarithm

In this section we recall the definition of the q-logarithm and prove some properties of it. Recall (cf. [21] for more on q-mathematics) that the q-analog of the integer $n \in \mathbb{Z}$ is defined to be

$$[n]_q := \frac{q^n - 1}{q - 1} \in \mathbb{Z}[q^{\pm 1}].$$

If $n \geq 1$, then we can rewrite

$$[n]_q = 1 + q + \ldots + q^{n-1}$$

and then the q-number actually lies in $\mathbb{Z}[q]$. For $n \geq 0$ we moreover get the relation

$$[-n]_q = \frac{q^{-n} - 1}{q - 1} = q^{-n}\frac{1 - q^n}{q - 1} = -q^{-n}[n]_q. \tag{2}$$

The q-numbers satisfy some basic relations, for example

$$[n + k]_q = q^k[n]_q + [k]_q \tag{3}$$

for $n, k \in \mathbb{Z}$, or

$$[m]_q = \frac{(q^n)^k - 1}{q^n - 1}\frac{q^n - 1}{q - 1} = \frac{(q^n)^k - 1}{q^n - 1}[n]_q,$$

if $n|m$. As further examples of q-analogs let us define the q-factorial for $n \geq 1$ as

$$[n]_q! := [1]_q \cdot [2]_q \cdot \ldots \cdot [n]_q \in \mathbb{Z}[q]$$

(with the convention that $[0]_q! := 1$) and, for $0 \leq k \leq n$, the q-binomial coefficient as

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Lemma 4.1. 1) *For $0 \leq k \leq n$ the q-binomial $\binom{n}{k}_q \in \mathbb{Z}[q]$.*

2) *For* $1 \leq k \leq n$ *the analog*

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q$$

of Pascal's identity holds.

Proof. 1) follows from 2) using induction and the easy case $\binom{n}{0}_q = 1$. Then 2) can be proved as follows: Let $1 \leq k \leq n$, then

$$\begin{aligned} q^k \tbinom{n-1}{k}_q + \tbinom{n-1}{k-1}_q &= \tfrac{[n-1]_q!}{[k-1]_q![n-1-k]_q!}\left(\tfrac{q^n}{[k]_q} + \tfrac{1}{[n-k]_q}\right) \\ &= \tfrac{[n-1]_q!}{[k-1]_q![n-1-k]_q!}\left(\tfrac{q^k[n-k]_q+[k]_q}{[k]_q[n-k]_q}\right. \\ &= \tfrac{[n-1]_q!}{[k-1]_q![n-1-k]_q!}\tfrac{[n]_q}{[k]_q[n-k]_q} \\ &= \tbinom{n}{k}_q \end{aligned}$$

using the addition rule (Equation (3)). □

Let us define a generalized q-Pochhammer symbol by

$$(x, y; q)_n := (x + y)(x + yq) \dots (x + yq^{n-1}) \in \mathbb{Z}[q^{\pm}1, x, y]$$

for $n \geq 1$ (setting $x = 1$ and $y := -a$ recovers the known q-Pochhammer symbol

$$(a; q)_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1}) = (1, -a; q)_n).$$

Moreover we make the convention

$$(x, y; q)_0 := 1.$$

In the q-world the generalized q-Pochhammer symbol replaces the polynomial

$$(x + y)^n.$$

For example one can show (using Lemma 4.1) the following q-binomial formula

$$(x, y; q)_n = \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k}_q x^{n-k} y^k. \tag{4}$$

Let us now come to q-derivations. We recall that the q-derivative $\nabla_q f$ of some polynomial $f \in \mathbb{Z}[q^{\pm 1}][x^{\pm 1}]$ is defined by

$$\nabla_q f(x) := \frac{f(qx) - f(x)}{qx - x} \in \mathbb{Z}[q^{\pm 1}][x^{\pm 1}].$$

Thus for example, if $f(x) = x^n$, $n \in \mathbb{Z}$, then we can calculate

$$\nabla_q(x^n) = \frac{q^n x^n - qx}{qx - x} = \frac{q^n - 1}{q - 1} x^{n-1} = [n]_q x^{n-1}.$$

The q-derivative satisfies an analog of the Leibniz rule, namely

$$\nabla_q(f(x)g(x)) = \nabla_q(f(x))g(qx) + f(x)\nabla_q(g(x)).$$

Similarly to the classical rule

$$\nabla_x((x + y)^n) = n\nabla_x((x + y)^{n-1})$$

we obtain the following relation for the generalized q-Pochhammer symbol.

Lemma 4.2. *Let $\nabla_q := \nabla_{q,x}$ denote the q-derivative with respect to x. Then the formula*

$$\nabla_q((x,y;q)_n) = [n]_q(x,y;q)_{n-1}$$

holds.

Proof. We proceed by induction on n. Let $n=1$. Then $(x,y;q)_n = x+y$ and

$$\nabla_q((x+y)) = 1.$$

Now let $n \geq 2$. We calculate using induction

$$\begin{array}{rcc} \nabla_q((x,y;q)_n) & = & \nabla_q((x,y;q)_{n-1}(x+yq^{n-1})) \\ & = & (x,y,q)_{n-1}\nabla_q(x+yq^{n-1}) + (qx-q^{n-1})\nabla_q((x,y;q)_{n-1}) \\ & = & (x,y;q)_{n-1}\cdot 1 + q(x-q^{n-2})[n-1]_q(x,y;q)_{n-2} \\ & = & (1+q[n-1]_q)(x,y;q)_{n-1} \\ & = & [n]_q(x-1)^{[n-1]_q} \end{array}$$

where we used the q-Leibniz rule and (Equation (3)). □

Similar as the polynomials

$$1, x-1, \frac{(x-1)^2}{2!}, \ldots, \frac{(x-1)^n}{n!}, \ldots$$

are useful for developing some function into a Taylor series around $x=1$ (because the derivative of one polynomial is the previous one) the q-polynomials

$$1, (x,-1;q)_1, \frac{(x,-1;q)_2}{[2]_q!}, \ldots, \frac{(x,-1;q)_n}{[n]_q!}, \ldots$$

are useful for developing a q-polynomial into some "q-Taylor series" around $x=1$. However, for this to make sense we have to pass to suitable completions and localize at $\{[n]_q\}_{n\geq 1}$. Let us be more precise about this. The $(q-1,x-1)$-completion $\mathbb{Z}[[q-1,x-1]]$ of $\mathbb{Z}[q,x]$ contains expressions of the form

$$\sum_{n=0}^{\infty} a_n(x,-1;q)_n$$

with $a_n \in \mathbb{Z}[[q-1]]$ because

$$(x,-1;q)_n = (x-1)(x-1+1-q)\ldots(x-1+(1-q)\frac{1-q^{n-1}}{1-q}) \in (q-1,x-1)^n.$$

Finally, the next calculations will take place in the ring

$$R_q := \mathbb{Z}[[q-1,\widehat{x-1]][1}/[n]_q|n\geq 1]_{(q-1,x-1)}$$

because

$$\frac{(x,-1;q)_n}{[n]_q!} \in (q-1,x-1)_{R_q}.$$

The ring R_q admits still surjection to

$$R_q \to \mathbb{Q}[[x-1]]$$

with kernel generated by $q-1$. Similarly, there is a morphism

$$\mathrm{ev}_1 \colon R_q \to \mathbb{Z}[[q-1]]\widehat{[1/[n]}_q|n\geq 1]_{(q-1)}$$

with kernel generated by $x-1$. Finally, the q-derivative ∇_q extends to a q-derivation on R_q and it induces the usual derivative after modding out $q-1$. We denote by ∇_q^n the n-fold decomposition of ∇_q and by

$$f(x)_{|x=1} := \mathrm{ev}_1(f(x))$$

the evaluation at $x=1$ of an element $f \in R_q$.

Lemma 4.3. *Let $f(x) \in R_q$. If $\nabla_q^n(f(x))_{|x=1} = 0$ for all $n \geq 0$, then $f(x) = 0$.*

Proof. As ∇_q reduces to the usual derivative modulo $q-1$, we see that f must be divisible by $q-1$, i.e., we can write $f(x) = (q-1)g(x)$ with $g(x) \in R_q$. But then $\nabla_q^n(g(x))_{x=1} = 0$ for all $n \geq 0$ and we can conclude as before that $q-1|g(x)$ which in the end implies

$$f(x) \in \bigcap_{k=1}^{\infty} (q-1)^k = \{0\}$$

because R_q is $(q-1)$-adically separated. □

Now we can state the q-Taylor expansion around $x=1$ for elements in R_q.

Proposition 4.4. *For any $f(x) \in R_q$ there is the Taylor expansion*

$$f(x) = \sum_{n=0}^{\infty} \nabla_q^n(f(x))_{|x=1} \frac{(x,-1;q)_n}{[n]_q!}.$$

Proof. Because

$$\nabla_q(\frac{(x,-1;q)_n}{[n]_q!}) = \frac{(x,-1;q)_{n-1}}{[n-1]_q!}$$

we can directly calculate that both sides have equal higher derivatives at $x=1$. Thus they agree by Lemma 4.3. □

Using this we can in Lemma 4.6 motivate the following formula for the q-logarithm.

Definition 4.5. We define the q-logarithm as

$$\log_q(x) := \sum_{n=1}^{\infty} (-1)^{n-1} q^{-n(n-1)/2} \frac{(x,-1;q)_n}{[n]_q} \in R_q.$$

In the ring R_q the element x is invertible, as

$$\frac{1}{x} = \frac{1}{1-(1-x)} = 1 + (1-x) + (1-x)^2 + \dots .$$

The q-derivative of the q-logarithm is $1/x$, similarly to the usual logarithm.

Lemma 4.6. *The q-logarithm $\log_q(x)$ is the unique $f(x) \in R_q$ satisfying $f(1) = 0$ and $\nabla_q(f(x)) = \frac{1}{x}$. Moreover,*

$$\log_q(x) = \frac{q-1}{\log(q)} \log(x).$$

Proof. That $\log_q(x)$ has q-derivative $1/x$ can be checked using Proposition 4.4 after writing $1/x$ in its q-Taylor expansion. Moreover, $\log_q(1) = 0$. For the converse pick f as in the statement. By Proposition 4.4 we can write

$$f(x) = \sum_{n=0}^{\infty} \nabla_q^n(f(x))_{|x=1} \frac{(x,-1;q)_n}{[n]_q!}.$$

and thus we have to determine

$$a_n := \nabla_q^n(f(x))_{|x=1}$$

for $n \geq 0$. By assumption we must have $a_0 = f(1) = 0$. Moreover, for $n \geq 1$

$$a_n = \nabla_q^n(f(x))_{|x=1} = \nabla_q^{n-1}(x^{-1})_{|x=1} = [-n+1]_q \dots [-1]_q.$$

Using $[-k]_q = -q^{-k}[k]_q$ for $k \in \mathbb{Z}$ the last expression simplifies to

$$[-n+1]_q \dots [-1]_q = (-1)^{n-1} q^{-n(n-1)/2} [n-1]_q!.$$

Thus we can conclude

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} q^{-n(n-1)/2} \frac{(x,-1;q)_n}{[n]_q} = \log_q(x).$$

For the last statement note that

$$f(x) := \frac{q-1}{\log(q)} \log(x)$$

exists in R_q (because $n \in R_q^\times$ for all $n \geq 1$) and satisfies $f(1) = 0$. Moreover,

$$\nabla_q(f(x)) = \frac{f(qx) - f(x)}{qx - x} = \frac{q-1}{\log(q)} \frac{\log(q) + \log(x) - \log(x)}{(q-1)x} = \frac{1}{x}$$

which implies $f(x) = \log_q(x)$ by the proven uniqueness of the q-logarithm. □

We now turn to prisms again. Define

$$\tilde{\xi} := [p]_q = 1 + q + \dots + q^{p-1}$$

and

$$\tilde{\xi}_r = \tilde{\xi}\varphi(\tilde{\xi}) \dots \varphi^{r-1}(\tilde{\xi})$$

for $r \geq 1$. Here, φ is the Frobenius lift on $\mathbb{Z}[q^{\pm 1}]$ satisfying $\varphi(q) = q^p$. Then $\tilde{\xi}$ is a distinguished element in the prism $\mathbb{Z}_p[[q-1]]$. The $\tilde{\xi}_r$ are again q-numbers, namely

$$\tilde{\xi}_r = [p^r]_q.$$

Let us recall the following situation from crystalline cohomology. Assume that A is a p-complete ring with an ideal $J \subseteq A$ equipped with divided powers

$$\gamma_n \colon J \to J, \ n \geq 1.$$

In this situation the logarithm

$$\log(x) := \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)! \gamma_n(x-1)$$

converges in A for every element $x \in 1 + J$. We now want to prove an analogous statement for the q-logarithm. Recall that for a prism (A, I) we defined the Nygaard filtration

$$\mathcal{N}^{\geq n} A := \{x \in A \mid \varphi(x) \in I^n\}, \ n \geq 0$$

in Definition 3.8. From now on, we assume that the prism (A, I) lives over $(\mathbb{Z}_q[[q-1]], (\tilde{\xi}))$. The expression

$$\gamma_{n,q}(x-y) := \frac{(x-y)(x-qy)\cdots(x-q^{n-1}y)}{[n]_q!}$$

is called the n-th q-divided power of $x-y$ (cf. [19, Rem. 1.4]). We will study the divisibility of

$$(x-y)(x-qy)\cdots(x-q^{n-1}y)$$

by

$$\tilde{\xi}, \varphi(\tilde{\xi}), \ldots.$$

The following statement is clear.

Lemma 4.7. *For $r \geq 1$ the polynomial (in q)*

$$\varphi^{r-1}(\tilde{\xi}) = \frac{q^{p^r}-1}{q^{p^{r-1}}-1}$$

is the minimal polynomial of a p^r-th root of unity ζ_{p^r}, i.e., the morphism

$$\mathbb{Z}[q]/(\varphi^{r-1}(\tilde{\xi})) \to \mathbb{Z}[\zeta_{p^r}],\ q \mapsto \zeta_{p^r}$$

is injective.

Thus reducing modulo $\varphi^{r-1}(\tilde{\xi})$ is the same as setting $q = \zeta_{p^r}$. Moreover, in $\mathbb{Z}[\zeta_{p^r}]$ there is the equality

$$z^{p^r} - 1 = \prod_{i=0}^{p^r-1} (z - \zeta_{p^r}^i).$$

Setting $z = \frac{x}{y}$ one thus arrives at the congruence

$$x^{p^r} - y^{p^r} \equiv (x-y)(x-qy)\cdots(x-q^{p^r-1}y) \bmod \varphi^{r-1}(\tilde{\xi}), \tag{5}$$

which will be useful.

Lemma 4.8. *Let $n \geq 1$ and for $r \geq 1$ write $n = a_r p^r + b_r$ with $a_r, b_r \geq 0$ and $b_r < p^r$. Then in $\mathbb{Z}_p[[q-1]]$*

$$[n]_q! = u \prod_{r \geq 1}^{\infty} \varphi^{r-1}(\tilde{\xi})^{a_r}$$

for some unit $u \in \mathbb{Z}_p[[q-1]]^\times$.

Proof. We may prove the statement by induction on n. Thus let us assume that it is true for $m = n-1$ and for $r \geq 1$ write $m = c_r p^r + d_r$ with $c_r, d_r \geq 0$ and $d_r < p^r$. If n is prime to p, then $[n]_q$ is a unit in $\mathbb{Z}_p[[q-1]]$ and it suffices to see that the righthand side is equal (up to some unit in $\mathbb{Z}_p[[q-1]]$) to

$$\prod_{r \geq 1}^{\infty} \varphi^{r-1}(\tilde{\xi})^{c_r}.$$

But n prime to p implies that $b_r > 0$ for all $r \geq 1$. Thus $c_r = a_r$ and $d_r = b_r - 1$, which implies that both products are equal. Now assume that p divides n and write $n = p^s n'$ with n' prime to p. Moreover, write $m = n-1 = c_r p^r + d_r$ as above.

Then we can conclude $a_r = c_r = 0$ for $r > s$ while $c_r = a_r - 1$ for $1 \leq r \leq s$ (as $d_r = p^r - 1$ for such r). Altogether we therefore arrive at

$$\begin{aligned} [n]_q! &= [n]_q[n-1]_q! \\ &= u'[n]_q \prod_{r\geq 1}^{\infty} \varphi^{r-1}(\tilde{\xi})^{c_r} \\ &= u'v \prod_{r\geq 1}^{\infty} \varphi^{r-1}(\tilde{\xi})^{a_r} \end{aligned},$$

$u' \in \mathbb{Z}_p[[q-1]]^\times$, where we used that

$$[n]_q = v[p^s]_q = v\varphi^{s-1}(\tilde{\xi})\dots\tilde{\xi}$$

for some unit $v \in \mathbb{Z}_p[[q-1]]$. □

Proposition 4.9. *Let (A, I) be a prism over $(\mathbb{Z}_p[[q-1]], (\tilde{\xi}))$ and let $x, y \in A$ be elements of rank 1 such that $\varphi(x-y) = x^p - y^p \in \tilde{\xi}A$. Then for all $n \geq 1$ the ring A contains a q-divided power*

$$\gamma_{n,q}(x-y) = \frac{(x-y)(x-qy)\cdots(x-q^{n-1}y)}{[n]_q!}$$

of $x-y$. Moreover, $\gamma_{n,q}$ lies in fact in the n-th step $\mathcal{N}^{\geq n}A$ of the Nygaard filtration of A.

Proof. Replacing A, x, y by the universal case we may assume that A is flat over $\mathbb{Z}_p[[q-1]]$. In particular, this implies that $\tilde{\xi}, \varphi(\tilde{\xi}), \dots$ are pairwise regular sequences (cf. Lemma 3.6). Fix $n \geq 1$. For $r \geq 1$ we write n as

$$n = a_r p^r + b^r$$

with $a_r, b_r \geq 0$ and $0 \leq b^r < p^r$. We claim that for each $r \geq 0$

$$\varphi^{r-1}(\tilde{\xi})^{a_r}$$

divides

$$(x-y)(x-qy)\cdots(x-q^{n-1}y).$$

This implies the proposition, namely by Lemma 4.8 we have

$$[n]_q! = u \prod_{r\geq 1} \varphi^{r-1}(\tilde{\xi})^{a_r}$$

for some unit $u \in A^\times$ while furthermore the morphism

$$A/([n]_q!) \to \prod_{r\geq 1} A/(\varphi^{r-1}(\tilde{\xi}))^{a_r}$$

is injective by Lemma 3.6. Thus fix $r \geq 1$. To prove our claim we may replace n by $n - b_r$ as

$$(x-y)(x-qy)\cdots(x-q^{n-b_r-1}y)$$

divides

$$(x-y)(x-qy)\cdots(x-q^{n-1}y).$$

Thus let us assume that $n = a_r p^r$. We claim that each of the following a_r many elements (note that their product is $(x-y)\cdots(x-q^{n-1}y)$)

$$\begin{gathered}(x-y)(x-qy)\cdots(x-q^{p^r-1}y),\\ (x-q^{p^r}y)(x-q^{p^r+1}y)\cdots(x-q^{2p^r-1}y),\\ \vdots\\ (x-q^{(a_r-1)p^r}y)(x-q^{(a_r-1)p^r+1}y)\cdots(x-q^{a_rp^{r+1}-1}y),\end{gathered}$$

is divisible by $\varphi^{r-1}(\tilde{\xi})$. For this recall the congruence (Equation (5))

$$x^{p^r} - y^{p^r} \equiv (x-y)(x-qy)\cdots(x-q^{p^r-1}y) \bmod \varphi^{r-1}(\tilde{\xi}).$$

Replacing in this congruence y by $q^{p^r}y, \ldots, q^{(a_r-1)p^r}y$ shows that each of the above a_r elements is congruent modulo $\varphi^{r-1}(\tilde{\xi})$ to an element of the form

$$x^{p^r} - q^k y^{p^r}$$

with $k \geq 0$ divisible by p^r. But we have

$$x^{p^r} - q^k y^{p^r} = (x^{p^r} - y^{p^r}) + y^{p^r}(1-q^k)$$

and we claim that under our assumptions both summands are divisible by $\varphi^{r-1}(\tilde{\xi})$. For the first summand we use that x, y are of rank 1 to write

$$x^{p^r} - y^{p^r} = \varphi^{r-1}(x^p - y^p) = \varphi^{r-1}(\tilde{\xi})\varphi^{r-1}(\frac{x^p - y^p}{\tilde{\xi}})$$

which makes sense as we assumed that

$$x^p - y^p \in \tilde{\xi}A.$$

For the second summand we note that

$$1 - q^k = \frac{1-q^k}{1-q^{p^r}}\varphi^{r-1}(\tilde{\xi})(1-q^{p^{r-1}})$$

with all factors in $\mathbb{Z}_p[[q-1]]$ as p^r divides k. It remains to prove that

$$\gamma_{n,q}(x-y) = \frac{(x-y)(x-qy)\cdots(x-q^{n-1}y)}{[n]_q!}$$

lies in $\mathcal{N}^{\geq n}A$. But

$$\varphi(\gamma_{n,q}) = \frac{(x^p-y^p)(x^p-q^py^p)\cdots(x^p-q^{p(n-1)}y^p)}{\varphi([n]_q!)}$$

and as we saw above $\tilde{\xi}$ divides each of the n factors

$$(x^p-y^p),\ (x^p-q^py^p),\ \cdots, (x^p-q^{p(n-1)}y^p).$$

But $\tilde{\xi}$ and $\varphi([n]_q!)$ form a regular sequence by Lemma 3.6 which implies that

$$(x^p-y^p)(x^p-q^py^p)\cdots(x^p-q^{p(n-1)}y^p)$$

is divisible by $\tilde{\xi}^n\varphi([n]_q)$ as was to be proven. This finishes the proof of the proposition. □

Moreover, we get the following lemma concerning the convergence of the q-logarithm.

Lemma 4.10. *Let (A, I) be a prism over $(\mathbb{Z}_p[[q-1]], (\tilde{\xi}))$. Then for every element $x \in 1 + \mathcal{N}^{\geq 1}A$ of rank 1 the series*

$$\log_q(x) = \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(n-1)/2} \frac{(x-1)(x-q)\cdots(x-q^{n-1})}{[n]_q}$$

is well-defined and converges in A. Moreover, $\log_q(x) \in \mathcal{N}^{\geq 1}A$ and

$$\log_q(x) \equiv x - 1 \bmod \mathcal{N}^{\geq 2}A.$$

Proof. By our assumption on x we get $\varphi(x-1) \in \tilde{\xi}A$ and thus we may apply Proposition 4.9 to $x = x$ and $y = 1$. Thus the q-divided powers

$$\gamma_{n,q}(x-1) = \frac{(x-1)(x-q)\cdots(x-q^{n-1})}{[n]_q!}$$

lie in A. Moreover, as

$$\log_q(x) = \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(n-1)/2} [n-1]_q! \gamma_{n,q}(x-1)$$

and the elements $[n-1]_q!$ tend to zero in A we can conclude that the series $\log_q(x)$ converges because A is $\tilde{\xi}$-adically complete. The claim concerning the Nygaard filtrations follows directly from $\gamma_{n,q}(x-1) \in \mathcal{N}^{\geq n}A$, which was proven in Proposition 4.9. □

5. Prismatic cohomology and topological cyclic homology

This section is devoted to the relation of the prismatic cohomology developed by Bhatt and Scholze [4] with topological cyclic homology (as described by Bhatt, Morrow and Scholze [3]) following [4, Section 11.5.].

Let R be a quasi-regular semiperfectoid ring (cf. [3, Definition 4.19.]), and let S be any perfectoid ring with a map $S \to R$.

Proposition 5.1. *The category of prisms (A, I) with a map $R \to A/I$ admits an initial object $(\Delta_R^{\mathrm{init}}, I)$, which is a bounded prism. Moreover, Δ_R^{init} identifies with the derived prismatic cohomology $\Delta_{R/A_{\mathrm{inf}}(S)}$, for any choice of S as before.*

Proof. See [4, Proposition 7.2, Proposition 7.10] or [1, Proposition 3.4.2]. □

In the following, we simply write $\Delta_R = \Delta_R^{\mathrm{init}} = \Delta_{R/A_{\mathrm{inf}}(S)}$.

Theorem 5.2. *Let R be a quasi-regular semiperfectoid ring. There is a functorial (in R) δ-ring structure on $\widehat{\Delta}_R^{\mathrm{top}} := \pi_0(\mathrm{TC}^-(R; \mathbb{Z}_p))$ refining the cyclotomic Frobenius. The induced map $\Delta_R = \Delta_R^{\mathrm{init}} \to \widehat{\Delta}_R^{\mathrm{top}}$ identifies $\widehat{\Delta}_R^{\mathrm{top}}$ with the completion with respect to the Nygaard filtration (Definition 3.8) of Δ_R, and is compatible with the Nygaard filtration on both sides.*

Proof. See [4, Theorem 11.10]. □

The Nygaard filtration on $\widehat{\Delta}_R^{\mathrm{top}}$ is defined as the double-speed abutment filtration for the (degenerating) homotopy fixed point spectral sequence

$$E_2^{ij} := H^i(\mathbb{T}, \pi_{-j}(\mathrm{THH}(R; \mathbb{Z}_p))) \Rightarrow \pi_{-i-j}(\mathrm{TC}^-(R; \mathbb{Z}_p))$$

for the $\mathbb{T} = S^1$-action on $\mathrm{THH}(R;\mathbb{Z}_P)$. If $v \in H^2(\mathbb{T},\mathbb{Z})$ is a generator, then multiplication by v induces isomorphisms

$$\pi_{2i}(\mathrm{TC}^-(R;\mathbb{Z}_p) \cong \mathcal{N}^{\geq i}\widehat{\mathbb{\Delta}}_R^{\mathrm{top}}$$

for $i \in \mathbb{Z}$.

Remark 5.3. We will only use the fact that $\widehat{\mathbb{\Delta}}_R$ is a prism in this paper (as we will apply the results of 3 to $\pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p)))$ and that the topological Nygaard filtration, defined via the homotopy fixed point spectral sequence, agrees with the Nygaard filtration from Definition 3.8, but the way one proves this is by showing the stronger statement that $\widehat{\mathbb{\Delta}}_R^{\mathrm{top}}$ is the Nygaard completion of $\mathbb{\Delta}_R$. We ignore if there is a more direct way to produce the δ-structure on $\widehat{\mathbb{\Delta}}_R$ (cf. [4, Remark 1.14.]).

We now discuss our choice of a generator of $H^2(\mathbb{T},\mathbb{Z})$. Consider the morphism $\mathbb{Z}[x] \to \mathbb{Z},\ x \mapsto 0$. The relative Hochschild homology

$$\mathrm{HH}(\mathbb{Z}/\mathbb{Z}[x]))$$

is concentrated in even degrees with

$$\pi_2(\mathrm{HH}(\mathbb{Z}/\mathbb{Z}[x])) \cong (x)/(x)^2.$$

This implies that the homotopy fixed point spectral sequence

$$E_2^{ij} = H^i(\mathbb{T}, \pi_{-j}(\mathrm{HH}(\mathbb{Z}/\mathbb{Z}[x])) \Rightarrow \pi_{-i-j}(\mathrm{HC}^-(\mathbb{Z}/\mathbb{Z}[x]))$$

degenerates. In particular, multiplication by a generator $w \in H^2(\mathbb{T},\mathbb{Z})$ induces an isomorphism

$$\pi_2(\mathrm{HH}(\mathbb{Z}/\mathbb{Z}[x]) \cong H^2(\mathbb{T}, \pi_2(\mathrm{HH}(\mathbb{Z}/\mathbb{Z}[x])))$$

where $H^2(\mathbb{T}, \pi_2(\mathrm{HH}(\mathbb{Z}/\mathbb{Z}[x])))$ identifies with

$$\mathcal{N}^{\geq 1}\pi_0(\mathrm{HC}^-(\mathbb{Z}/\mathbb{Z}[x]))/\mathcal{N}^{\geq 2}\pi_0(\mathrm{HC}^-(\mathbb{Z}/\mathbb{Z}[x]))$$

where we denoted by $\mathcal{N}^{\geq *}\pi_0(\mathrm{HC}^-(\mathbb{Z}/\mathbb{Z}[x]))$ the abutment filtration on $\pi_0(\mathrm{HC}^-(\mathbb{Z}/\mathbb{Z}[x]))$. By definition the $\mathbb{T}$-action on $\mathrm{HH}(\mathbb{Z}/\mathbb{Z}[x])$ is $\mathbb{Z}[x]$-linear which implies that $\pi_0(\mathrm{HC}^-(\mathbb{Z}/\mathbb{Z}[x]))$ is canonically a $\mathbb{Z}[x]$-algebra.

Lemma 5.4. *Via the natural morphism*

$$\mathbb{Z}[x]/(x)^2 \cong \pi_0(\mathrm{HC}^-(\mathbb{Z}/\mathbb{Z}[x]))/\mathcal{N}^{\geq 2}\pi_0(HC^-(\mathbb{Z}/\mathbb{Z}[x])).$$

Proof. This can be proved by a calculation in Hochschild homology (like in [14, Proposition 2.12]) or deduced from the well-known statement

$$\pi_0(\mathrm{HC}^-(\mathbb{F}_l/\mathbb{Z}_l))/\mathcal{N}^{\geq 2}\pi_0(\mathrm{HC}^-(\mathbb{F}_l/\mathbb{Z}_/)) \cong \mathbb{Z}_l/l^2$$

for all primes l as follows (cf. [14, Proposition 2.12]). Set

$$A := \pi_0(\mathrm{HC}^-(\mathbb{Z}/\mathbb{Z}[x]))/\mathcal{N}^{\geq 2}\pi_0(HC^-(\mathbb{Z}/\mathbb{Z}[x])).$$

From the degenerate spectral sequence

$$E_2^{ij} = H^i(\mathbb{T}, \pi_{-j}(\mathrm{HH}(\mathbb{Z}/\mathbb{Z}[x])) \Rightarrow \pi_{-i-j}(\mathrm{HC}^-(\mathbb{Z}/\mathbb{Z}[x]))$$

one deduces that there is an exact sequence

$$0 \to I \to A \to \mathbb{Z} \to 0$$

with $I \cong \mathbb{Z}$. The natural $\mathbb{Z}[x]$-algebra structure on A induces a morphism $(x)/(x)^2 \to I$ because $x \mapsto 0 \in \mathbb{Z}$ and $I^2 = 0$. If l is a prime then

$$A \otimes_{\mathbb{Z}[x]} \mathbb{Z}_l \cong \pi_0(\mathrm{HC}^-(\mathbb{F}_l/\mathbb{Z}_l)) \cong \mathbb{Z}_l/l^2$$

where $x \mapsto l \in \mathbb{Z}_l$. In particular, x maps to (a multiple of) $l \in I \otimes_{\mathbb{Z}[x]} \mathbb{Z}_l \cong \mathbb{Z}_l$. As this holds for all primes l we get that $(x)/(x^2) \cong I$ as desired. □

Definition 5.5. We define $w_{\mathrm{can}} \in H^2(\mathbb{T}, \mathbb{Z})$ to be the unique generator such that the multiplication

$$(x)/(x^2) \cong \pi_2(\mathrm{HH}(\mathbb{Z}/\mathbb{Z}[x])) \xrightarrow{w_{\mathrm{can}}} \pi_0(\mathrm{HC}^-(\mathbb{Z}/\mathbb{Z}[x]))$$

sends the class of x to $x \cdot 1_{\pi_0(\mathrm{HC}^-(\mathbb{Z}/\mathbb{Z}[x]))/\mathcal{N}^{\geq 2}\pi_0(\mathrm{HC}^-(\mathbb{Z}/\mathbb{Z}[x])))}$. Here $\cdot$ denotes the canonical $\mathbb{Z}[x]$-algebra structure on $\pi_0(\mathrm{HC}^-(\mathbb{Z}/\mathbb{Z}[x]))$.

Let R be a quasi-regular semiperfectoid ring and let $\theta\colon W(R^\flat) \to R$. Set

$$J := \ker(\theta\colon W(R^\flat) \to R).$$

We denote again by $w_{\mathrm{can}} \in H^2(\mathbb{T}, \pi_0(\mathrm{HH}(R;\mathbb{Z}))$ the image of the canonical generator $w_{\mathrm{can}} \in H^2(\mathbb{T}, \mathbb{Z})$ from 5.5 under the canonical morphism coming from $\mathbb{Z} \mapsto \pi_0(\mathrm{HH}(R;\mathbb{Z}))$.

Let $v \in \pi_{-2}(\mathrm{TC}^-(R;\mathbb{Z}_p))$ be a lift of the element $w_{\mathrm{can}} \in H^2(\mathbb{T}, \pi_0(\mathrm{HH}(R;\mathbb{Z}_p))$. The multiplication by v induce an isomorphism

$$\pi_2(\mathrm{TC}^-(R;\mathbb{Z}_p)) \cong \mathcal{N}^{\geq 1}\pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p)) \cong \mathcal{N}^{\geq 1}\widehat{\mathbb{\Delta}}_R$$

By construction this multiplication fits into a commutative diagram

$$\begin{tikzcd}
\pi_2(\mathrm{TC}^-(R;\mathbb{Z}_p)) \arrow[r, "v"] \arrow[d] & \pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p) \arrow[d] \\
\pi_2(\mathrm{HH}(R;\mathbb{Z}_p) \arrow[r, "w_{\mathrm{can}}"] & \pi_0(\mathrm{HC}^-(R;\mathbb{Z}_p)).
\end{tikzcd}$$

The ring $\pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p))$ is canonically a $W(R^\flat)$-algebra.[12]

The following lemma is immediate.

Lemma 5.6. *The multiplication by v (with v a lift of w_{can}) from $\pi_2(\mathrm{TC}^-(R;\mathbb{Z}_p)) \to \pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p)$ induces on*

$$J/J^2 \cong \pi_2(\mathrm{HH}(R;\mathbb{Z}_p)) \to \pi_0(\mathrm{HC}^-(R;\mathbb{Z}_p))/\mathcal{N}^{\geq 2}\pi_0(\mathrm{HC}^-(R;\mathbb{Z}_p))$$

the morphism which sends $j \in J$ to $j \cdot 1_{\pi_0(\mathrm{HC}^-(R;\mathbb{Z}_p)/\mathcal{N}^{\geq 2}\pi_0(\mathrm{HC}^-(R;\mathbb{Z}_p))}$ where $\cdot$ denotes the canonical $W(R^\flat)$-algebra structure on $\pi_0(\mathrm{HC}^-(R;\mathbb{Z}_p))$.

Proof. Consider the relative Hochschild homology $\mathrm{HH}(R/W(R^\flat))$ (where R is an $W(R^\flat)$-algebra via θ). Then

$$\mathrm{HH}(R;\mathbb{Z}_p) \cong \mathrm{HH}(R/W(R^\flat))$$

by 2.6. Fix $j \in J$. We obtain a morphism

$$\mathbb{Z}[x] \to W(R^\flat),\ x \mapsto j.$$

Then the claim follows from the definition of w_{can} using naturality. □

The following lemma clarifies the calculation of $\pi_*(\mathrm{TC}^-(R;\mathbb{Z}_p))$ for a perfectoid ring R (as done in [3, Proposition 6.2., Proposition 6.3.]) with the condition that v lifts the canonical generator w_{can}.

[12] One can use that $W(R^\flat) \cong \pi_0(\mathrm{TC}^-(R';\mathbb{Z}_p)$ for some suitable perfectoid ring R' mapping to R or that the canonical morphism $\pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p)) \to \pi_0(\mathrm{THH}(R;\mathbb{Z}_p) \cong R$ is a pro-nilpotent thickening of R. In both cases, the $W(R^\flat)$-algebra structure extends the morphism $\theta\colon W(R^\flat) \to R$.

Lemma 5.7. *Let R be a perfectoid ring and let ξ be a generator or $\ker(\theta\colon W(R^\flat) \to R)$. Let $u \in \pi_2(\mathrm{TC}^-(R;\mathbb{Z}_p))$ be a lift of the class of ξ in $\pi_2(\mathrm{THH}(R;\mathbb{Z}_p) \cong (\xi)/(\xi^2)$. Let $v \in \pi_{-2}(\mathrm{TC}^-(R;\mathbb{Z}_p))$ such that $uv = \xi \in \pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p)$. Then v lifts $w_{\mathrm{can}} \in H^2(\mathbb{T}, \pi_0(\mathrm{THH}(R;\mathbb{Z}_p)))$.*

Proof. Let $\tilde{v} \in \pi_2(\mathrm{TC}^-(R;\mathbb{Z}_p))$ be a lift of (the image of) the canonical generator $w_{\mathrm{can}} \in H^2(\mathbb{T}, \pi_0(\mathrm{THH}(R;\mathbb{Z}_p))$. By 5.6 we see that

$$\tilde{v}u$$

reduces to $\xi \in \pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p))/\mathcal{N}^{\geq 2}\pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p))$. In particular, $uv = u\tilde{v}$ modulo $\mathcal{N}^{\geq 2}$. This implies that v and $\tilde{v}$ agree in $H^2(\mathbb{T}, \pi_0(\mathrm{THH}(R;\mathbb{Z}_p))$. □

6. The p-completed cyclotomic trace in degree 2

Now we are settled to prove our main theorem on the identification of the p-completed cyclotomic trace. Let us fix the following notation. Set $\mathbb{Z}_p^{\mathrm{cycl}}$ as the p-completion of $\mathbb{Z}_p[\mu_{p^\infty}]$ and choose some p-power compatible system of p-power roots of unity

$$\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots) \in (\mathbb{Z}_p^{\mathrm{cycl}})^\flat$$

with $\zeta_p \neq 1$. This choice determines several elements as we will now discuss. Set

$$q := [\varepsilon]_\theta \in A_{\mathrm{inf}}(\mathbb{Z}_p^{\mathrm{cycl}}) := W((\mathbb{Z}_p^{\mathrm{cycl}})^\flat) \cong \pi_0(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p)),$$

$$\mu := q - 1,$$

$$\tilde{\xi} := [p]_q = \frac{q^p - 1}{q - 1} = 1 + q + \ldots + q^{p-1}.$$

and

$$\xi = \varphi^{-1}(\tilde{\xi}).$$

We now carefully construct elements

$$u \in \pi_2(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p)),$$

$$v \in \pi_{-2}(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p))$$

such that $uv = \xi \in \pi_0(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p))$.[13] The elements u, v will be uniquely determined by ε. Let

$$\mathrm{ctr}\colon \mathbb{Z}_p(1)(\mathbb{Z}_p^{\mathrm{cycl}}) \to \pi_2(\mathrm{TC}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p))$$

be the cyclotomic trace in degree 2. We denote by the same symbol the composition

$$\mathrm{ctr}\colon \mathbb{Z}_p(1)(\mathbb{Z}_p^{\mathrm{cycl}}) \to \pi_2(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p))$$

with the canonical morphism $\mathrm{TC}(-;\mathbb{Z}_p) \to \mathrm{TC}^-(-;\mathbb{Z}_p)$. Let

$$\mathrm{can}\colon \mathrm{TC}^-(-;\mathbb{Z}_p) \to \mathrm{TP}(-;\mathbb{Z}_p)$$

be the canonical morphism (from homotopy to Tate fixed points).

Lemma 6.1. *The element*

$$\mathrm{can}(\mathrm{ctr}(\varepsilon^{-1})) \in \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p))$$

is divisible by μ.

[13] We need a finer statement than [3, Proposition 6.2. and Proposition 6.3.] which asserts the existence of some u, v as above with $uv = a\xi$ for some unspecified unit $a \in A_{\mathrm{inf}}(\mathbb{Z}_p^{\mathrm{cycl}})^\times$.

A similar statement (in terms of TF) is proven in [12, Proposition 2.4.2.] (cf. [13, Definition 4.1.]) using the explicit description of the cyclotomic trace in degree 1 via TR from [11, Lemma 4.2.3.].

Proof. Fix a generator

$$\sigma' \in \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p)).$$

It suffices to show that $\mathrm{can} \circ \mathrm{ctr}(\varepsilon)$ maps to 0 under the composition

$$\pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p)) \xrightarrow{\sigma'} \pi_0(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p)) \cong A_{\mathrm{inf}}(\mathbb{Z}_p^{\mathrm{cycl}}) \to W(\mathbb{Z}_p^{\mathrm{cycl}})$$

because the kernel of $A_{\mathrm{inf}}(\mathbb{Z}_p^{\mathrm{cycl}}) \to W(\mathbb{Z}_p^{\mathrm{cycl}})$ is generated by μ (cf. [2, Lemma 3.23.]). It therefore suffices to prove the statement for $\mathcal{O}_C$ for $C/\mathbb{Q}_p^{\mathrm{cycl}}$ an algebraically closed, complete non-archimedean extension. Over $\mathcal{O}_C$ we can (after changing σ') find

$$u' \in \pi_2(\mathrm{TC}^-(\mathcal{O}_C;\mathbb{Z}_p)),$$
$$v' \in \pi_{-2}(\mathrm{TC}^-(\mathcal{O}_C;\mathbb{Z}_p))$$

such that

$$u'v' = \xi = \frac{\mu}{\varphi^{-1}(\mu)},$$
$$\mathrm{can}(v') = \sigma^{-1}$$

and the cyclotomic Frobenius maps u' to σ'. Then multiplication by v induces an isomorphism

$$\pi_2(\mathrm{TC}(\mathcal{O}_C;\mathbb{Z}_p)) \cong A_{\mathrm{inf}}(\mathcal{O}_C)^{\varphi=\tilde{\xi}}.$$

By [9, Proposition 6.2.10.]

$$(A_{\mathrm{inf}}(\mathcal{O}_C)[1/p])^{\varphi=\tilde{\xi}}$$

is 1-dimensional over $\mathbb{Q}_p$ and thus generated by μ (as $\mu \neq 0$ and $\varphi(\mu) = \tilde{\xi}\mu$). But μ is not divisible by p in $A_{\mathrm{inf}}(\mathcal{O}_C)$ as it maps to a unit in $W(C)$. This proves that $A_{\mathrm{inf}}(\mathcal{O}_C)^{\varphi=\tilde{\xi}} = \mathbb{Z}_p\mu$, which implies the claim. □

It follows from 6.4 that in fact

$$\mathrm{ctr}(\varepsilon^{-1}) = \mu.$$

Let us define

$$\sigma := \frac{\mathrm{ctr}(\varepsilon^{-1})}{\mu} \in \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p))$$

and

$$u := \xi\sigma \in \pi_2(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}})).$$

More precisely, the element u is defined via $\mathrm{can}(u) = \xi\sigma$ (note that $\xi\sigma$ lies indeed in the image of

$$\mathrm{can}\colon \pi_2(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p)) \to \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p))$$

as the abutment filtration for the Tate fixed point spectral sequence on $\pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p))$ is the ξ-adic filtration.

Lemma 6.2. *The element u defined above lifts the class of*

$$\xi \in \pi_2(\mathrm{THH}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p)) \cong \pi_2(\mathrm{HH}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p) \cong (\xi)/(\xi^2).$$

Proof. By definition

$$\mathrm{can}(u) = \frac{\xi}{\mu}\mathrm{ctr}(\varepsilon^{-1}) \in \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p)).$$

Now

$$\frac{\xi}{\mu} = \frac{1}{\varphi^{-1}(\mu)}$$

and $(\xi)/(\xi^2)$ is $\varphi^{-1}(\mu)$-torsion free as a module over $A_{\mathrm{inf}}(\mathbb{Z}_p^{\mathrm{cycl}})$ (because

$$\theta(\varphi^{-1}(\mu)) = \zeta_p - 1 \neq 0 \in \mathbb{Z}_p^{\mathrm{cycl}}).$$

Moreover, by construction the cyclotomic trace lifts the Dennis trace in Hochschild homology (this, together with the fact that its target is $\mathrm{TC}(-)$, is the only property of the cyclotomic trace we need, cf. [5, Section 10], [6, Section 5]). Thus by Proposition 2.5,

$$\mathrm{ctr}(\varepsilon^{-1}) \equiv [\varepsilon] - 1 \in (\xi)/(\xi^2).$$

Thus

$$u \equiv \frac{[\varepsilon]-1}{\varphi^{-1}(\mu)} = \xi \in (\xi)/(\xi^2)$$

as desired. □

In particular, we see that the element

$$\sigma \in \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p))$$

is a generator. Set

$$v := \sigma^{-1} \in \pi_{-2}(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p)) \overset{\mathrm{can}}{\cong} \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p)).$$

Then

$$uv = \xi.$$

Moreover, one has the following (important) additional property (which, up to changing ξ by some unit, is implied by the conjunction of [3, Proposition 6.2., Proposition 6.3.]).

Lemma 6.3. *The cyclotomic Frobenius*

$$\varphi^{h\mathbb{T}} \colon \pi_2(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p)) \to \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p))$$

sends u to σ.

Proof. The cyclotomic Frobenius $\varphi^{h\mathbb{T}}$ is linear over the Frobenius on A_{inf}. Thus we can calculate (note $\frac{\xi}{\mu} = \varphi^{-1}(\mu)$)

$$\varphi^{h\mathbb{T}}(u) = \varphi(\frac{\xi}{\mu})\varphi^{h\mathbb{T}}(\mathrm{ctr}(\varepsilon^{-1})) = \frac{1}{\mu}\varphi^{h\mathbb{T}}(\mathrm{ctr}(\varepsilon^{-1})).$$

But

$$\varphi^{h\mathbb{T}}(\mathrm{ctr}(\varepsilon^{-1})) = \mathrm{can}(\mathrm{ctr}(\varepsilon^{-1}))$$

as the cyclotomic trace has image in $\pi_2(\mathrm{TC}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p))$. This implies that $\varphi^{h\mathbb{T}}(u) = \frac{\mathrm{ctr}(\varepsilon^{-1})}{\mu} = \sigma$ as desired. □

By Lemma 6.3 one can conclude that there is a commutative diagram, whose vertical arrows are isomorphisms,

$$\begin{tikzcd}
\pi_2(\mathrm{TC}(R;\mathbb{Z}_p)) \arrow[r] \arrow[d, "\alpha"] & \pi_2(\mathrm{TC}^-(R;\mathbb{Z}_p)) \arrow[r, "\varphi^{h\mathbb{T}}-\mathrm{can}"] \arrow[d, "v"] & \pi_2(\mathrm{TP}(R;\mathbb{Z}_p)) \arrow[d, "\sigma^{-1}"] \\
\widehat{\mathbb{\Delta}}_R^{\varphi=\tilde{\xi}} \arrow[r] & \mathcal{N}^{\geq 1}\widehat{\mathbb{\Delta}}_R \arrow[r, "\frac{\varphi}{\tilde{\xi}}-1"] & \widehat{\mathbb{\Delta}}_R
\end{tikzcd}$$

for any quasi-regular semiperfectoid $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra R. We remind the reader that the induced isomorphism

$$\alpha\colon \pi_2(\mathrm{TC}(R;\mathbb{Z}_p)) \cong \widehat{\mathbb{\Delta}}_R^{\varphi=\tilde{\xi}}$$

depends only on ε.

For a quasi-regular semiperfectoid ring R we denote by

$$[-]_{\tilde{\theta}}\colon R^\flat = \varprojlim_{x\mapsto x^p} R \to \mathbb{\Delta}_R$$

the Teichmüller lift. More precisely, the canonical morphism $R \to \overline{\mathbb{\Delta}}_R$ induces a morphism $\iota\colon R^\flat \to \overline{\mathbb{\Delta}}_R^\flat$ and $[-]_{\tilde{\theta}}$ is the composition of ι with the Teichmüller lift for the surjection

$$\mathbb{\Delta}_R \to \overline{\mathbb{\Delta}}_R.$$

We set

$$[-]_\theta := [(-)^{1/p}]_{\tilde{\theta}}.$$

We will consider the p-adic Tate module

$$T_pR^\times = \varprojlim_{n\geq 0} R^\times[p^n]$$

of $R^\times$ as being embedded into $R^\flat$ as the elements with first coordinate equal to 1.

We are ready to state and prove our main theorem.

Theorem 6.4. *Let R be a quasi-regular semiperfectoid $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Then the composition*

$$T_pR^\times \to \pi_2(K(R;\mathbb{Z}_p)) \xrightarrow{\mathrm{ctr}} \pi_2(\mathrm{TC}(R;\mathbb{Z}_p)) \overset{\alpha}{\cong} \widehat{\mathbb{\Delta}}_R^{\varphi=\tilde{\xi}}$$

is given by sending $x \in T_p(R^\times)$ to

$$\log_q([x^{-1}]_\theta) = \sum_{n=1}^{\infty}(-1)^{n-1}q^{n(n-1)/2}\frac{([x^{-1}]_\theta - 1)([x^{-1}]_\theta - q)\cdots([x^{-1}]_\theta - q^{n-1})}{[n]_q}.$$

Proof. Replacing R by the universal case $\mathbb{Z}_p^{\mathrm{cycl}}\langle x^{1/p^\infty}\rangle/(x-1)$ we may assume that R is p-torsion free and (thus) that $(\widehat{\mathbb{\Delta}}_R, (\tilde{\xi}))$ is transversal (by 3.3 it suffices to see that (p,ξ) is a regular sequence which follows as $\widehat{\mathbb{\Delta}}_R/\xi \cong \widehat{L\Omega_{R/\mathbb{Z}_p^{\mathrm{cycl}}}}$, by [3, Theorem 7.2.(5)], is p-torsion free).

Let us define

$$\mathrm{ctr}_2\colon T_pR^\times \to \pi_2(K(R;\mathbb{Z}_p)) \xrightarrow{\mathrm{ctr}} \pi_2(\mathrm{TC}(R;\mathbb{Z}_p)).$$

By Theorem 5.2 the canonical morphism

$$\iota\colon \mathbb{\Delta}_R \to \pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p))$$

is compatible with the Nygaard filtrations and identifies $\pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p))$ with the Nygaard completion $\widehat{\mathbb{\Delta}}_R$ of $\mathbb{\Delta}_R$. By 3.10 the morphism

$$\mathbb{\Delta}^{\pi=\tilde{\xi}} \hookrightarrow \mathcal{N}^{\geq 1}\mathbb{\Delta}_R/\mathcal{N}^{\geq 2}\mathbb{\Delta}_R \cong \mathcal{N}^{\geq 1}\widehat{\mathbb{\Delta}}_R/\mathcal{N}^{\geq 2}\widehat{\mathbb{\Delta}}_R$$

is injective. Hence it suffices to show that the two morphisms $\log_q([-]_\theta)$ and $\alpha \circ \mathrm{ctr}$ agree modulo $\mathcal{N}^{\geq 2}\widehat{\mathbb{\Delta}}_R$. Multiplication by the element $v \in \pi_{-2}(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p))$ constructed after 6.2 induces an isomorphism

$$J/J^2 \cong \pi_2(\mathrm{THH}(R;\mathbb{Z}_p)) \overset{v}{\cong} \mathcal{N}^{\geq 1}\widehat{\mathbb{\Delta}}/\mathcal{N}^{\geq 2}\widehat{\mathbb{\Delta}}_R$$

where J is the kernel of the surjection

$$\theta\colon W(R^\flat) \to R.$$

By 5.6, 5.7 and 6.2 this isomorphism sends the class of $j \in J$ to $j \cdot 1_{\widehat{\mathbb{\Delta}}_R/\mathcal{N}^{\geq 2}\widehat{\mathbb{\Delta}}_R}$ for the canonical $W(R^\flat)$-algebra structure on

$$\begin{aligned}\widehat{\mathbb{\Delta}}_R/\mathcal{N}^{\geq 2}\widehat{\mathbb{\Delta}}_R &\cong \pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p))/\mathcal{N}^{\geq 2}\pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p)) \\ &\cong \pi_0(\mathrm{HC}^-(R/W(R^\flat)))/\mathcal{N}^{\geq 2}\pi_0(\mathrm{HC}^-(R/W(R^\flat)))\end{aligned}$$

(which lifts the morphism θ). Let $x \in \mathbb{Z}_p(1)(R)$. By 4.10

$$\log_q([x]_\theta) \equiv [x]_\theta - 1 \bmod \mathcal{N}^{\geq 2}\widehat{\mathbb{\Delta}}_R.$$

On the other hand, as the cyclotomic trace reduces to the Dennis trace Dtr, we can calculate using 2.5 and 5.6

$$\begin{aligned}\alpha(\mathrm{ctr}(x)) &\equiv v\mathrm{Dtr}(x) \\ = v([x^{-1}]_\theta - 1) &= ([x^{-1}]_\theta - 1) \cdot 1_{\widehat{\mathbb{\Delta}}_R/\mathcal{N}^{\geq 2}\widehat{\mathbb{\Delta}}_R} \bmod \mathcal{N}^{\geq 2}\widehat{\mathbb{\Delta}}_R\end{aligned}$$

Thus we can conclude

$$\log_q([x^{-1}]_\theta) = \alpha \circ \mathrm{ctr}(x)$$

as desired. □

Corollary 6.5. *Let R be a quasi-regular semiperfectoid $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. The map*

$$\log_q([-]_\theta)\colon T_p(R^\times) \to \widehat{\mathbb{\Delta}}_R^{\varphi=\tilde{\xi}}$$

is a bijection.

Proof. Since both sides satisfy quasi-syntomic descent[14], one can assume, as in [3, Proposition 7.17], that R is w-local and such that $R^\times$ is divisible. In this case, the map

$$T_p(R^\times) \to \pi_2(K(R;\mathbb{Z}_p))$$

is a bijection. Moreover, [7, Corollary 6.9] shows that

$$\mathrm{ctr} : \pi_2(K(R;\mathbb{Z}_p)) \to \pi_2(\mathrm{TC}(R;\mathbb{Z}_p))$$

is also bijective. As by Theorem 6.4, the composite of these two maps is the map $\log_q([(-)^{1/p}]_{\tilde{\theta}})$, this proves the corollary. □

[14]For $T_p(-)^\times$ this follows from p-completely faithfully flat descent on p-complete rings with bounded p^∞-torsion, cf. [1, Appendix], for $\widehat{\mathbb{\Delta}}_R^{\varphi=\tilde{\xi}}$ this is is proven in [3].

Remark 6.6. As explained at the end of the introduction, one can give a direct and more elementary proof of Corollary 6.5 when R is the quotient of a perfect ring by a finite regular sequence ([22]) or when R is a p-torsion free quotient of a perfectoid ring by a finite regular sequence and p is odd. But we do not know how to prove it directly in general.

References

[1] Johannes Anschütz and Arthur-César Le Bras. Prismatic Dieudonné theory, Preprint 2019.

[2] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Integral p-adic Hodge theory. *Publications mathématiques de l'IHÉS*, 128(1):219–397, 2018.

[3] Bhargav. Bhatt, Matthew Morrow, and Peter Scholze. Topological Hochschild homology and integral p-adic Hodge theory. *ArXiv e-prints*, February 2018.

[4] Bhargav Bhatt and Peter Scholze. Prisms and prismatic cohomology. available at `http://www.math.uni-bonn.de/people/scholze/Publikationen.html`.

[5] Andrew J Blumberg, David Gepner, and Gonçalo Tabuada. A universal characterization of higher algebraic k-theory. *Geometry & Topology*, 17(2):733–838, 2013.

[6] Marcel Bökstedt, Wu Chung Hsiang, and Ib Madsen. The cyclotomic trace and algebraic K-theory of spaces. *Inventiones mathematicae*, 111(1):465–539, 1993.

[7] Dustin Clausen, Akhil Mathew, and Matthew Morrow. K-theory and topological cyclic homology of henselian pairs. *arXiv e-prints*, page arXiv:1803.10897, March 2018.

[8] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy. *The local structure of algebraic K-theory*, volume 18. Springer Science & Business Media, 2012.

[9] Laurent Fargues and Jean-Marc Fontaine. Courbes et fibrés vectoriels en théorie de Hodge p-adique. preprint on webpage at `https://webusers.imj-prg.fr/~laurent.fargues/Prepublications.html`.

[10] Ofer Gabber. K-theory of henselian local rings and henselian pairs. *Contemp. Math*, 126:59–70, 1992.

[11] Thomas Geisser and Lars Hesselholt. Topological cyclic homology of schemes. In *Proceedings of Symposia in Pure Mathematics*, volume 67, pages 41–88. Providence, RI; American Mathematical Society; 1998, 1999.

[12] Lars Hesselholt. On the topological cyclic homology of the algebraic closure of a local field. In *An alpine anthology of homotopy theory*, volume 399 of *Contemp. Math.*, pages 133–162. Amer. Math. Soc., Providence, RI, 2006.

[13] Lars Hesselholt. Topological Hochschild homology and the Hasse-Weil zeta function. In *An alpine bouquet of algebraic topology*, volume 708 of *Contemp. Math.*, pages 157–180. Amer. Math. Soc., Providence, RI, 2018.

[14] Achim Krause and Thomas Nikolaus. Lectures on topological hochschild homology and cyclotomic spectra. available at `https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/papers.html`.

[15] Jean-Louis Loday. *Cyclic homology*, volume 301. Springer Science & Business Media, 2013.

[16] J Peter May and Kate Ponto. *More concise algebraic topology: localization, completion, and model categories*. University of Chicago Press, 2011.

[17] Thomas Nikolaus. The group completion via localizations of ring spectra. Available at `https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/papers.html`.

[18] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. *arXiv e-prints*, page arXiv:1707.01799, Jul 2017.

[19] Jon P. Pridham. On q-de Rham cohomology via λ-rings. *Mathematische Annalen*, pages 1–28, 2016.

[20] Daniel Quillen. Higher algebraic K-theory: I. In *Higher K-theories*, pages 85–147. Springer, 1973.

[21] Peter Scholze. Canonical q-deformations in arithmetic geometry. *Ann. Fac. Sci. Toulouse Math. (6)*, 26(5):1163–1192, 2017.

[22] Peter Scholze and Jared Weinstein. Moduli of p-divisible groups. *Camb. J. Math.*, 1(2):145–237, 2013.

[23] The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`, 2017.

[24] Robert W Thomason. Algebraic K-theory and étale cohomology. In *Annales scientifiques de l'École Normale Supérieure*, volume 18, pages 437–552, 1985.

[25] Robert W Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift*, pages 247–435. Springer, 1990.

[26] Charles A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K-theory.

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