Let F be a function field over a finite field:
$\exists X$ smooth projective geom connected curve $X / k = \mathbb{F}_q$
s.t. $F = k(X)$.

Let $\mathbb{A}_F$ = ring of adèles of $F = \prod'_{x \in |X|} F_x$.
$\cup\!|$
$\mathbb{O} = \prod_{x \in X} \mathcal{O}_x$.

From the arithmetic perspective, function fields behave like number fields; in particular, the whole Langlands program also makes sense for them.

Let $n \geq 1$, $\ell \neq p$ prime.

<u>Th</u> (unramified global Langlands)
Let $\sigma : \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_n(\overline{\mathbb{Q}_\ell})$ which is:
• continuous • everywhere unramified • geom irreducible.
Then, $\exists!$ (up to scalar in $\overline{\mathbb{Q}_\ell}^\times$) non-zero
$f_\sigma : GL_n(F) \backslash GL_n(\mathbb{A}_F) / GL_n(\mathbb{O}) \to \overline{\mathbb{Q}_\ell}$, which is

• cuspidal
[for every non-trivial partition $\underline{n} = (n_1, \dots, n_r)$ of n, associated parabolic $P_{\underline{n}}$ with unipotent $V_{\underline{n}}$, $\forall$ Haar measure on $V_{\underline{n}}(F) \backslash V_{\underline{n}}(\mathbb{A}_F)$,
$$\int_{V_{\underline{n}}(F) \backslash V_{\underline{n}}(\mathbb{A}_F)} f(mg)\, dm = 0.]$$

• an Hecke eigenfunction for σ:
$\forall x \in |X|$, $\forall i = 1, \dots, n$, $T_x^i(f_\sigma) = q_x^{-i(i-1)/2} \operatorname{tr}(\Lambda^i \sigma(\mathrm{Frob}_x)) . f_\sigma$

$$g \mapsto \int_{GL_n(\mathcal{O}_x) \begin{pmatrix} \pi_x & & & \\ & \ddots & & \\ & & \pi_x & \\ & & & \ddots \end{pmatrix} GL_n(\mathcal{O}_x)} f_\sigma(gh)\, dh$$

normalized Haar measure on $GL_n(F_x)$.

[More conceptually: $\forall x$, $\mathcal{H}_x \simeq \mathrm{Rep}(GL_n, \overline{\mathbb{Q}_\ell})$
$\overline{\mathbb{Q}_\ell} \cdot f_\sigma \hookrightarrow \chi_{f_\sigma, x}$ character of $\mathcal{H}_x$
Condition $\Leftrightarrow$ $\chi_{f_\sigma, x} \leftrightarrow$ character $\mathrm{Rep}(GL_n, \overline{\mathbb{Q}_\ell}) \to \overline{\mathbb{Q}_\ell}$
$[V] \mapsto \operatorname{tr}(\sigma(\mathrm{Frob}_x), V)$]

How to construct f_σ? In fact, there is a natural candidate for f_σ.
To explain it, fix $\omega \in \Omega^1_{F/k} \setminus \{0\}$. ②

& $\psi : k \to \overline{\mathbb{Q}}_\ell^\times$ character, $\neq 1$.

Let $\Psi : F \setminus \mathbb{A}_F \to \overline{\mathbb{Q}}_\ell^\times$, $(a_x)_{x\in|X|} \mapsto \psi\left(\sum_{x\in|X|} \mathrm{tr}_{k(x)/k}(\mathrm{Res}(a_x\omega))\right)$

All characters of $F\setminus\mathbb{A}_F$ are of the form

$\Psi_\gamma : y \mapsto \Psi(\gamma y)$, for some $\gamma \in F$ (self-duality of $\mathbb{A}_F$)

Let's momentarily assume $n=2$.

Let $f : GL_2(F)\setminus GL_2(\mathbb{A}_F) \to \overline{\mathbb{Q}}_\ell$ smooth.

For each $g \in GL_2(\mathbb{A}_F)$, can write ~~for~~ the Fourier expansion of

$$N(F)\setminus N(\mathbb{A}_F) \simeq F\setminus\mathbb{A}_F \to \overline{\mathbb{Q}}_\ell$$

upper-unipotent

$$n \mapsto f(ng)$$

Get: $\forall g \in GL_2(\mathbb{A}_F)$ $\forall n' \in N(\mathbb{A}_F)$

$$f(n'g) = \sum_{\gamma\in F}\left(\int_{N(F)\setminus N(\mathbb{A}_F)} f(ng)\,\Psi^{-1}(\gamma n)\,dn\right)\Psi(\gamma n').$$

Take $n'=1$, and assume f cuspidal:

$$f(g) = \sum_{\gamma\in F^\times}\left(\int_{N(F)\setminus N(\mathbb{A}_F)} f(ng)\,\psi^{-1}(\gamma n)\,dn\right)$$

i.e. $f(g) = \sum_{\gamma\in F^\times} W_{f,\psi}\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\cdot g\right)$ with

$$W_{f,\psi} : g \mapsto \int_{N(F)\setminus N(\mathbb{A}_F)} f(ng)\,\psi^{-1}(n)\,dn$$

It lives in $C^\infty(GL_2(\mathbb{A}_F))^{(N(\mathbb{A}_F),\Psi)} = \left\{\begin{array}{l}\text{smooth functions}\\ W : GL_2(\mathbb{A}_F) \to \overline{\mathbb{Q}}_\ell\\ \text{s.t. } W(ng) = \Psi(n)W(g)\end{array}\right\}$
$\forall n \in N(\mathbb{A}_F), g \in GL_2(\mathbb{A}_F)$

Back to general n: with more work, can prove (induction)

Let $M = \begin{pmatrix} GL_{n-1} & * \\ 0 & 1\end{pmatrix} \subseteq GL_n$ mirabolic subgp

Prop (Shalika): $\exists\ GL_n(\mathbb{A}_F)$-eq isom:

$$\Phi : C^\infty(GL_n(\mathbb{A}_F))^{(N(\mathbb{A}_F),\psi)} \simeq C^\infty_{cusp}(M(F)\backslash GL_n(\mathbb{A}_F))$$

$$W \longmapsto \sum_{\gamma \in N_{n-1}(F)\backslash GL_{n-1}(F)} W\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} -\right)$$

So far, everything works without unramifiedness hypotheses. But:

Th (Casselman-Shalika) $\forall x \in |X|$, $\forall \gamma$ conjugacy class in $GL_n(\overline{\mathbb{Q}_\ell})$, $\exists$ explicit function $W_{\gamma,x} : GL_n(F_x) \to \overline{\mathbb{Q}_\ell}$

- right $GL_n(\mathcal{O}_x)$-inv
- $\forall n \in N(F_x)\ \forall g \in GL_n(F_x)$, $W_{\gamma,x}(ng) = \psi(\mathrm{tr}_{k(x)/k}(\mathrm{Res}(n\omega)))\, W_{\gamma,x}(g)$
- $\forall i = 1, \dots, n$,
$$T^i_x(W_{\gamma,x}) = q_x^{-\frac{i(i-1)}{2}}\, \mathrm{tr}(\Lambda^i \gamma)\, W_{\gamma,x}.$$
- $W_{\gamma,x}(1) = 1$.

Moreover, unique with these properties.

Let $\sigma : \mathrm{Gal}(\overline{F}/F) \to GL_n(\overline{\mathbb{Q}_\ell})$, everywhere unramified.

Set $W_\sigma : GL_n(\mathbb{A}_F) \to \overline{\mathbb{Q}_\ell}$

$$g \longmapsto \prod_{x \in |X|} W_{\sigma(\mathrm{Frob}_x),x}(g_x)$$

and $f'_\sigma = \Phi(W_\sigma)$

$f'_\sigma \in C_{cusp}(M(F)\backslash GL_n(\mathbb{A}_F)/GL_n(\mathcal{O}), \overline{\mathbb{Q}_\ell})$

& $\forall x \in |X|$, $\forall i = 1, \dots, n$, $T^i_x(f'_\sigma) = q_x^{-\frac{i(i-1)}{2}}\, \mathrm{tr}(\Lambda^i \mathrm{Frob}_x))\, f'_\sigma$

Moreover, unique with these properties.

So the question really becomes: why is f'_σ left $GL_n(F)$-inv?
Surprisingly hard to prove directly ...

Idea (Deligne, Drinfeld, Laumon) Replace functions by sheaves.
 $n=1$, $n=2$, $n \geq 2$

Recall functions-sheaves dictionary: Z scheme/stack over $k = \mathbb{F}_q$

$K \in D^b_{et}(Z, \overline{\mathbb{Q}_\ell}) \rightsquigarrow \mathrm{tr}_K : Z(k) \to \overline{\mathbb{Q}_\ell}$

$$x \longmapsto \sum_i (-1)^i\, \mathrm{tr}(F_k, \mathcal{H}^i(K)_x)$$

Basic observations:

- σ everywhere unramified $\longleftrightarrow$ rank n $\overline{\mathbb{Q}}_\ell$-local system $\mathbb{L}$ on X $\left(\pi_1(X) = \mathrm{Gal}(\overline{F}/F)^{\mathrm{unr}}\right)$
- $\mathrm{Bun}_n(k) \simeq GL_n(F)\backslash GL_n(\mathbb{A}_F)/GL_n(\mathcal{O})$ (Weil)

  set of iso classes of rank n vector bundles on X

Goal: find f_σ as trace-function of $\mathrm{Aut}_{\mathbb{L}} \in D_{\acute{e}t}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$

moduli stack of rk n vb on X

Two main steps:

1) • Construct $\mathrm{Aut}'_{\mathbb{L}} \in D_{\acute{e}t}(\mathrm{Bun}'_n, \overline{\mathbb{Q}}_\ell)$

   moduli-stack of pairs $(\mathcal{E}, s)$, $\mathcal{E}$ rk n vb, $s: \omega_X \hookrightarrow \mathcal{E}$

   s.t. $f'_\sigma = \mathrm{tr}_{\mathrm{Aut}'_{\mathbb{L}}}$.

2) • Prove that $\mathrm{Aut}'_{\mathbb{L}}$ descends to $\mathrm{Aut}_{\mathbb{L}} \in D_{\acute{e}t}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ by exploiting the geometry of $\mathrm{Bun}'_n \to \mathrm{Bun}_n$

Rest of today's talk: focus on Step 1). $\longrightarrow$ ENDED HERE ON OCT. 15.

The ℓ-adic Fourier transform

S scheme/stack over $k = \mathbb{F}_q$.

$\mathcal{V} \to S$ (geometric) vector bundle, $\mathcal{V}^\vee \to S$ dual v.b.

$\psi: k \to \overline{\mathbb{Q}}_\ell^\times$ as before gives rise to $\mathcal{L}_\psi$ local system on $\mathbb{A}^1_k$. (Artin-Schreier seq)

$$\begin{tikzcd} & \mathcal{V}^\vee \times_S \mathcal{V} \arrow[dl, "\check{\pi}"'] \arrow[dr, "\pi"] \arrow[r, "\alpha"] & \mathbb{A}^1_k \\ \mathcal{V}^\vee \arrow[dr] & & \mathcal{V} \arrow[dl] \\ & S & \end{tikzcd}$$

Define:

$$\mathcal{F}_{\psi, \mathcal{V}\to\mathcal{V}^\vee}: D_{\acute{e}t}(\mathcal{V}, \overline{\mathbb{Q}}_\ell) \to D_{\acute{e}t}(\mathcal{V}^\vee, \overline{\mathbb{Q}}_\ell)$$
$$A \mapsto R\check{\pi}_!(\pi^* A \otimes \alpha^* \mathcal{L}_\psi)$$

Deligne/Laumon/Verdier: it is an equivalence (involutive) commuting with Verdier duality, preserving perverse sheaves.

Induces classical FT by taking traces.

Drinfeld-Laumon's construction:

Let $i \geq 0$, Coh_i: alg. stack of flat coh sheaves on X over the base of generic rank i.

Can define $\mathcal{E}_i \subseteq \mathrm{Coh}_i$ open substack s.t.
if $\mathcal{E}_i$ universal coh sheaf on $\mathcal{E}_i \times_k X$, then
$\mathcal{V}_i = \underline{\mathrm{Hom}}(\omega_X^{\otimes i}, \mathcal{E}_i)$ is a vector bundle over $\mathcal{E}_i$
with dual vector bundle $\mathcal{V}_i^\vee = \underline{\mathrm{Ext}}^1(\mathcal{E}_i, \omega_X^{\otimes i+1})$.

(Serre duality)

Have:

$$\mathcal{V}_i^\vee \times_{\mathcal{E}_i} \mathcal{V}_i \xrightarrow{\alpha} \mathrm{Ext}^1(\omega_X^{\otimes i}, \omega_X^{\otimes i+1}) \simeq \mathbb{A}^1_k$$

$$\begin{tikzcd}[column sep=small] & \mathcal{V}_i^\vee \times_{\mathcal{E}_i} \mathcal{V}_i \arrow[dl] \arrow[dr] & \\ \mathcal{V}_i^\vee \arrow[dr] & & \mathcal{V}_i \arrow[dl] \\ & \mathcal{E}_i & \end{tikzcd}$$

Let $\mathcal{F}_{\psi,i} := \mathcal{F}_{\psi, \mathcal{V}_i \to \mathcal{V}_i^\vee}$: $D_{et}(\mathcal{V}_i, \overline{\mathbb{Q}}_\ell) \to D_{et}(\mathcal{V}_i^\vee, \overline{\mathbb{Q}}_\ell)$.

Set: $\mathcal{V}_i^\circ \subseteq \mathcal{V}_i$ $= \{(\mathcal{E}, s),\ s \text{ injective}\}$, $\mathcal{V}_i^{\vee,\circ} \subseteq \mathcal{V}_i^\vee$

$\{0 \to \mathcal{O} \to \mathcal{E}' \to \mathcal{E} \to 0 \text{ s.t. } \mathcal{E}' \in \mathcal{E}_{i+1}\}$

Note: $\mathcal{V}_i^{\vee,\circ} \overset{\gamma_i}{\simeq} \mathcal{V}_{i+1}^\circ$

$$(0 \to \omega_X^{i+1} \to \mathcal{E}' \to \mathcal{E} \to 0) \mapsto (\omega_X^{i+1} \hookrightarrow \mathcal{E}')$$

Obtain:

$$\begin{tikzcd}[column sep=small] \mathcal{V}_2^\vee \arrow[ddr] & \mathcal{V}_2 \arrow[dd] & \supseteq \mathcal{V}_2^\circ & \overset{\simeq}{\underset{\gamma_1}{\leftarrow}} \mathcal{V}_1^{\vee,\circ} & \underset{j_1^\vee}{\subseteq} \mathcal{V}_1^\vee \arrow[dr] & & \mathcal{V}_1 \arrow[dl] & \underset{j_1}{\supseteq} \mathcal{V}_1^\circ \arrow[r, "f"] & \mathrm{Coh}_0 \\ & & & & & \mathcal{E}_1 & & & \\ & \mathcal{E}_2 & & & & & & & \end{tikzcd}$$

$$(\omega_X \hookrightarrow \mathcal{E}) \mapsto \mathcal{E}/\omega_X$$

and functors: $\mathcal{F}^\circ_{\psi,i}$: $D_{et}(\mathcal{V}_i^\circ, \overline{\mathbb{Q}}_\ell) \to D_{et}(\mathcal{V}_{i+1}^\circ, \overline{\mathbb{Q}}_\ell)$

$$:= \gamma_i^* \circ j_i^{\vee *} \circ \mathcal{F}_{\psi,i} \circ j_{i,!}$$

For $\sigma \hookrightarrow \mathbb{L} \in D_{et}(X, \overline{\mathbb{Q}}_\ell)$, construct
$\mathcal{L}_\mathbb{L} \in D_{et}(\mathrm{Coh}_0, \overline{\mathbb{Q}}_\ell)$ (generalization of Casselman-Shalika)

and then define:

$$\mathrm{Aut}'_\mathbb{L} := (\mathcal{F}^\circ_{\psi,n-1} \circ \cdots \circ \mathcal{F}^\circ_{\psi,1})(f^*\mathcal{L}_\mathbb{L}).$$