ADIC SPACES & THE FARGUES-FONTAINE CURVE



How to do geometry over non-archimedean fields? A definition of "p-adic manifolds" imitating the classical definition in differential geometry runs into the problem that the topology is totally disconnected.

After Tate, Berkovich, Huber proposed a general theory dealing with this difficulty.

Convention: A non-archimedean (NA) field is a field $K$ endowed with a multiplicative valuation $|.| : K^* \to \mathbb{R}_{>0}$.

**Def**: A topological ring $A$ is Huber if $A$ admits an open subring $A_0 \subseteq A$ which is $I$-adic, for some f.g. ideal $I \subset A_0$ (i.e., $(I^n)_{n \geq 0}$ form a basis of neighb'ds of $0$)

If $A$ contains a topologically nilp. unit, $A$ is said to be Tate. $=:$ pseudo-uniformizer (p.u.)

Ex:
not Tate
- Any discrete ring is Huber. ($A_0 = A$, $I = (0)$)
- Any adic ring for a f.g. ideal $I$ is Huber ($A_0 = A$, $I = I$).

Tate
- $K$ complete NA field, A $K$-Banach algebra is Huber. ($A_0 =$ unit ball, $I = (\pi)$, $\pi \in K^*$, $|\pi| < 1$)
  e.g. $A = K\langle T\rangle := (K^\circ[T])^{\wedge\pi}[\frac{1}{\pi}]$.

**Def**: A Huber, resp. Tate, pair is a pair $(A, A^+)$, with $A$ Huber ring, resp. Tate ring, and $A^+ \subseteq A^\circ$ open int. closed and $A^+ \subseteq A^\circ$.

**Def**: A continuous valuation on a top ring $A$ is
$$|.| : A \to \Gamma \cup \{0\}$$
← tot. ordered ab grp

s.t.: $|1| = 1$, $|0| = 0$, $|ab| = |a||b|$, $|a+b| \leq \max(|a|, |b|)$ and, $\forall \gamma \in \Gamma$, $\{a \in A, |a| < \gamma\}$ open.

Two such are equivalent if for $\Gamma, \Gamma'$ chosen minimal, $\Gamma \simeq \Gamma'$ and cg diagram.

<u>Def</u>: $(A, A^+)$ Huber pair.

Let $\mathrm{Spa}(A, A^+) = \{$ continuous valuations $A$ s.t. $|A^+| \le 1 \}$
with top generated by opens of the form $\{x, \ |f(x)| \le |g(x)| \neq 0\}$ for $f, g \in A$

(here $|f(x)|$ = notation for $x(f)$).

Huber: This top space is <u>spectral</u> (in particular, quasi-compact).

<u>Rk</u>: There is a con<u>tinuous</u> map $\Phi : \mathrm{Spa}(A, A^+) \longrightarrow \mathrm{Spec}(A)$
$$x \longmapsto \mathrm{supp}(x)$$

(continuity: if $f \in A$, $\Phi^{-1}(D(f)) = \{x, \ 0 = |0(x)| \le |f(x)| \neq 0\} = U\left(\frac{0}{f}\right)$.)

<u>Ex 1</u>: $\mathrm{Spa}\,\mathbb{Z} := \mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$

$$\begin{array}{c} \mathrm{Spa}\,\mathbb{Z} \\ \downarrow \Phi \\ \mathrm{Spec}\,\mathbb{Z} \end{array} \qquad \begin{array}{ccc} \eta_{triv} & & \\ \downarrow & & \\ \eta_p & \rightsquigarrow & s_p \\ \downarrow & & \downarrow \\ (0) & \rightsquigarrow & (p) \end{array}$$

~~the maps~~

$\eta_{triv}$: sends everything non-zero to 1.

$\eta_p : \mathbb{Z} \to \mathbb{Z}_p \xrightarrow{|\cdot|_p} \mathbb{R}_{>0}$ ($p$ prime)

$s_p : \mathbb{Z} \to \mathbb{Z}/p \longrightarrow \{0, 1\}$

triv valuation

<u>Ex 2</u>: $K$ complete alg closed NA field. If $r \in \mathbb{R}_{\ge 0}$, write
$D(x, r) = \{y \in K, \ |y - x| \le r\}$, $\mathring{D}(x, r) = \{y \in K, \ |y - x| < r\}$.

Let $A = K\langle T\rangle$, $A^+ = K^\circ\langle T\rangle$. Description of the points of $X = \mathrm{Spa}(A, A^+)$:

(1) "Classical points": $x \in K^\circ = D(0, 1)$
$f \mapsto |f(x)|$ $\longleftrightarrow$ max ideals of $A$.

(2), (3): "Points of the limbs": $0 \le r < 1$, $x \in K^\circ$.

(2): $r \in |K^\times|$ (3): $r \notin |K^\times|$

$$x_r : f = \sum a_n (T - x)^n \mapsto \sup_{y \in D(x,r)} |f(y)| = \sup_n |a_n| r^n.$$

Points of type (2) are branching points.

[ (4) "Dead ends": $(D_i)_{i \ge 0}$ decreasing seq of disks with $\cap D_i = \emptyset$
$$f \mapsto \inf_i \sup_{y \in D_i} |f(y)|$$ ]

not essential: non-spherical completeness of $K$.

(5) $x \in k^\circ$, $\quad 0 < r \le 1$.

$\Gamma_r = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$, with $r' < \gamma < r \quad \forall r' < r$

$x_{<r} : f = \sum a_n (T-x)^n \longmapsto \sup_n |a_n| \gamma^n$

(only depends on $\mathring{D}(x,r)$).

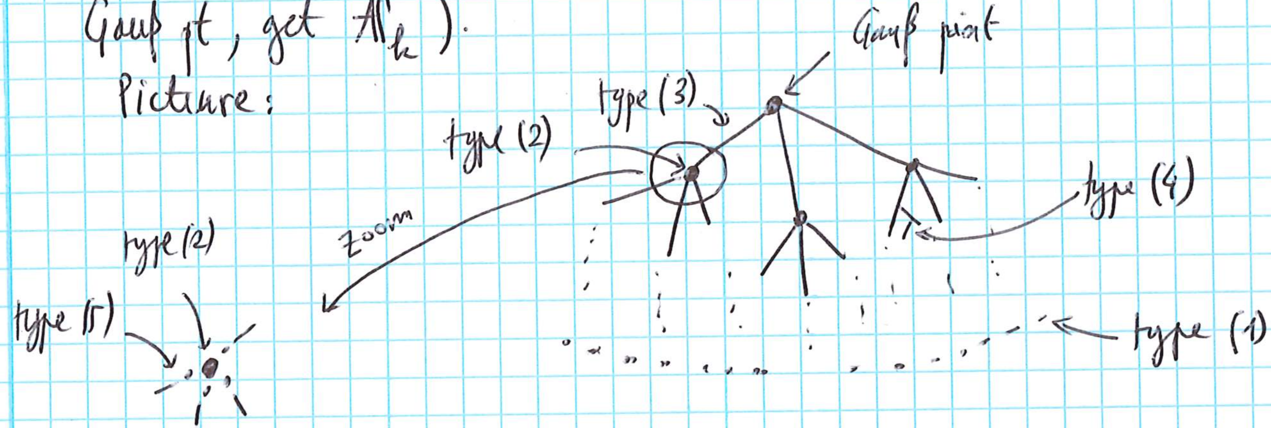
Similarly: $x_{>r}$, which only depends on $D(x,r)$.

If $r \notin |k^\times|$, $x_{<r} = x_{>r} = x_r$.

Note: $x_{>1} \notin X$.

Note: All points are closed, except type (2) points; closure of a type (2) point is type (5) points around it $\simeq \mathbb{P}^1_k$ (except for Gauß pt, get $\mathbb{A}^1_k$).

Picture:




Rk: 1) Krull dimension of $X$ is 1.

2) Let $U = \{ |T(x)| = 1 \} \subset X$ open

$V = \cancel{\{|T(x)| < 1\}}$
$\bigcup_{\varepsilon > 0} \{ |T(x)| \le 1 - \varepsilon \}$ open.

But $U \cup V \subsetneq X$!

Indeed, $x_{<1} \notin U \cup V$. Thus $X$ isn't disconnected.

Rk: Specialization of points on adic spaces arise in two ways:

"vertical ~~specialization~~ generalization"
- $x : A \to \Gamma \cup \{0\}$, $H \subset \Gamma$ convex subgp

  Define $x_{/H} : A \to \Gamma/H \cup \{0\}$

  $f \longmapsto \begin{cases} |f(x)| \bmod H & \text{if } |f(x)| \neq 0 \\ 0 & \text{otherwise.} \end{cases}$

"horizontal specialization"
- $x : A \to \Gamma \cup \{0\}$, $H \le \Gamma$ subgp (with some properties I won't specify)

  Define $x_{|H} : A \to \Gamma \cup \{0\}$

  $f \longmapsto \begin{cases} |f(x)| & \text{if } |f(x)| \in H \\ 0 & \text{otherwise} \end{cases}$

("vertical / horizontal": wrt to $\Phi$) (consider the example of Spa $\mathbb{Z}$)

**Def:** let $s \in A$, $T \subset A$ finite subset s.t. $TA \subseteq A$ open.
Let $U\left(\frac{T}{s}\right) = \{x \in X, \ |t(x)| \leq |s(x)| \neq 0, \ \forall t \in T\}$
"rational subset".

**Th** (Huber) let $U = U\left(\frac{T}{s}\right) \subset X = \mathrm{Spa}(A, A^+)$ rational subset.
$\exists$ complete Huber pair $(A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X(U)^+)$
s.t. $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to \mathrm{Spa}(A, A^+)$ factors over $U$ and is universal for such maps. Moreover, homeomorphism onto $U$
($\Rightarrow U$ spectral).

[ Pf sketch: $(B, B^+)$ s.t.

$$\begin{tikzcd} \mathrm{Spa}(B,B^+) \arrow[r] \arrow[dr] & \mathrm{Spa}(A,A^+) \\ & U \arrow[u] \end{tikzcd}$$

- $s$ invertible in $B$
  $\Rightarrow A[\frac{1}{s}] \to B$
- $\frac{t}{s}$, $t \in T$, are $|\cdot| \leq 1$ everywhere in $B \Rightarrow \frac{t}{s} \in B^+$.
- can choose ring of def $A_0, B_0$ s.t. $\frac{t}{s} \in B_0 \ \forall t \in T$ and get
  $\underbrace{A_0\left[\frac{t}{s}, t \in T\right]}_{\text{endow with the } I A_0[\frac{t}{s}, t]\text{-top.}} \to B_0$.

(skip if necessary)

Key: defines a ring top on $A[\frac{1}{s}]$ s.t. $A_0[\frac{t}{s}, t \in T]$ ring of def.
Want multiplication by $\frac{1}{s}$ continuous, i.e. $\frac{1}{s} I^n \subset A_0[\frac{t}{s}, t]$ for $n \gg 0$

$I^n \subseteq TA_0$ for $n \gg 0$ (true since $TA_0$ open) $\Longrightarrow$

Then set $(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = \left(A[\frac{1}{s}], \text{int. closure of } A^+[\frac{t}{s}, t \in T]\right)^\wedge$ ]

Rk: Rational subsets are spectral, hence qc, but not the opens defining the top on $X$: in general, if $A$ Tate with p.u. $\varpi$,

$$X\left(\frac{f}{g}\right) = \bigcup_n U\left(\frac{f, \varpi^n}{g}\right)$$

**Def:** For $W \subseteq X$ open, set $\mathcal{O}_X^{(+)}(W) = \varprojlim_{\substack{U \subset W \\ \text{rational subset}}} \mathcal{O}_X^{(+)}(U)$.
Structure sheaf (in practice always a sheaf...)

If $x \in X$, let $\mathcal{O}_{X,x}^{(+)} = \varinjlim_{\substack{U \ni x \\ \text{open}}} \mathcal{O}_X^{(+)}(U)$ (colimit as rings)

<u>Prop</u>: 1) The valuation $x : f \mapsto |f(x)|$ extends to $\mathcal{O}_{X,x}$
and $\mathcal{O}^+_{X,x} = \{ f \in \mathcal{O}_{X,x}, \ |f(x)| \leq 1 \}$.

2) The stalk $\mathcal{O}_{X,x}$ is local, max ideal
$m_x = \{ f, \ |f(x)| = 0 \}$. Residue field $k(x)$.

3) The stalk $\mathcal{O}^+_{X,x}$ is local, too, max ideal
$\{ f \in \mathcal{O}^+_{X,x}, \ |f(x)| < 1 \}$.

4) The valuation of $\mathcal{O}_{X,x}$ endows $k(x)$ with the structure of a ~~NA~~ valued field with valuation ring $k(x)^+$ = image of $\mathcal{O}^+_{X,x}$ in $k(x)$.

5) Let $\Sigma_A = \{ (A, A^+) \xrightarrow{\varphi} (k, k^+)$ $k$ NA field complete and $\varphi(A) \subset k$ generates a dense subfield $\}$

Then $\Sigma_A \longrightarrow \mathrm{Spa}(A, A^+)_{an}$
$\varphi \longmapsto$ image of closed pt in $\mathrm{Spa}(k, k^+)$ under $\mathrm{Spa}(\varphi)$
is bijective.

Tate case

6) If $A$ Tate, $\mathrm{Spa}(A, A^+) = \mathrm{Spa}(A, A^+)_{an}$ and all specializations are <u>vertical specializations</u> (keep $k$ fixed, change $k^+$)

7) If $A$ Tate, p.u. $\varpi$, the ring $\mathcal{O}^+_{X,x}$ is $\varpi$-adically henselian and the natural map
$(\mathcal{O}^+_{X,x})^{\wedge\varpi} \longrightarrow (k(x)^+)^{\wedge\varpi}$ is an isomorphism.

8) $A$ Tate: The pairs $(\mathcal{O}_{X,x}, m_x)$, $(\mathcal{O}^+_{X,x}, m_x)$ are henselian.

<u>Rk</u>: Points (6)-(8) show that adic spaces attached to Tate pairs behave differently than schemes:

- Set of generalizations of a pt is <u>totally ordered</u> (never happens for schemes of dim $> 1$)
- 7(8) $\Rightarrow$ Zariski closed subsets are cofiltered limits of rational open subsets!
- 7) + 8) $\Rightarrow$ $\underset{U \ni x}{2\text{-}\lim}\ \mathcal{O}_X(U)_{f\acute{e}t} \simeq k(x)_{f\acute{e}t}$.

Rk: In Example 2, • type (1),(3),(4): $\overline{k(x)} = k$, $\mathrm{rk}(\Gamma_x) = 1$

$\overline{(\ )}$ = res. field

• type (2): $\overline{k(x)} \neq k$. Then $\overline{k(x)}/k$ has trans deg 1, $\Gamma_x = \Gamma_K$.

• type (5): $\mathrm{rk}(\Gamma_x) > 1$, $\overline{k(x)} = k$.

~~More examples~~

Def: Adic space: object of the category of triples $(X, \mathcal{O}_X, \{v_x\}_{x\in X})$ locally of the form $\mathrm{Spa}(A, A^+)$, $(A, A^+)$ Huber pair (sheafy).

$X$: top space; $\mathcal{O}_X$: sheaf of top. rings s.t. $(X, \mathcal{O}_X)$ loc. ringed space (ignoring the top); $v_x$: cont. val on $\mathcal{O}_{X,x}$

Examples of adic spaces:

• The closed disk $\mathbb{D} = \mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z}[T])$ represents the sheaf $\mathcal{O}^+$.
If $K$ complete NA field, $\mathbb{D} \times_{\mathrm{Spa}\,\mathbb{Z}} \mathrm{Spa}(K, K^\circ) \cong \mathbb{D}_K := \mathrm{Spa}(K\langle T\rangle, K^\circ\langle T\rangle) = \mathbb{D}_K(1)$

• The Adic affine line: $\mathbb{A}^1 = \mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z})$ Represents $\mathcal{O}$.
If $K$ complete NA field, $\mathbb{A}^1 \times_{\mathrm{Spa}\,\mathbb{Z}} \mathrm{Spa}(K, K^\circ) \simeq \mathbb{A}^1_K$
$$\mathbb{A}^1_K = \bigcup_n \underbrace{\mathrm{Spa}(K\langle \varpi^n T\rangle, K^\circ\langle \varpi^n T\rangle)}_{\mathbb{D}_K(|\varpi|^{-n})}$$

Rk: Fiber products of adic spaces are a bit funny (and do not always exist): behave well when morphisms are adic.

Rk: $\mathbb{D}_K \subset \mathbb{A}^1_K$ open & $\overline{\mathbb{D}}_K = \mathbb{D}_K \cup \{\text{rk 2 pt}\}$
$= \mathrm{Spa}(K\langle T\rangle, K^\circ + TK^{\circ\circ}\langle T\rangle)$

• The open unit disk $\mathring{\mathbb{D}} = \mathrm{Spa}\,\mathbb{Z}[[T]]$
If $K$ complete NA field,
$$\mathring{\mathbb{D}} \times_{\mathrm{Spa}\,\mathbb{Z}} \mathrm{Spa}(K, K^\circ) \simeq \mathring{\mathbb{D}}_K := \bigcup_n \mathbb{D}_K(p^{-1/n}).$$

- The Fargues–Fontaine curve:

Let $E$ local field, res. field $k = \mathbb{F}_q$, uniformizer $\pi$.

Let $(R, R^+)$ Tate pair $/\mathbb{F}_q$.

Would like to make sense of "$\mathrm{Spa}(E, \mathcal{O}_E) \times_{\mathrm{Spa}\,\mathbb{F}_q} \underbrace{\mathrm{Spa}(R, R^+)}_{= S}$"

Can do it if $\mathrm{char}(E) = p$ i.e.

$$E = \mathbb{F}_q((\pi))$$

Then:

$$\mathrm{Spa}\, \mathcal{O}_E \times_{\mathrm{Spa}\,\mathbb{F}_q} \cancel{\mathrm{Spa}(R,R^+)} = \mathrm{Spa}\, \underbrace{R^+[[\pi]]}_{(\pi, \varpi)\text{-adic top}}$$

so

$$\mathrm{Spa}\, E \times S = (\mathrm{Spa}\, R^+[[\pi]]) \setminus V(\pi[\varpi]) = \mathbb{D}^*_S.$$

In general, observe that if $A$ perfect, $\exists!$ (up to $\simeq$) lift $\tilde{A}/\mathcal{O}_E$ flat, $\pi$-adically complete s.t. $\tilde{A}/\pi \simeq A$.

One choice is $\tilde{A} = W_{\mathcal{O}_E}(A) := W(A) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$.

Thus, can set $Y_{S,E} := (\mathrm{Spa}\, W_{\mathcal{O}_E}(R^+)) \setminus V(\pi[\varpi])$.

The Frobenius $\phi_S$ acts properly discontinuously on $Y_{S,E}$, so can define: $X_{S,E} := Y_{S,E} / \phi_S^{\mathbb{Z}}$.