$v \mapsto \lim \tilde{x}_n^{p^n}$

The Fargues-Fontaine curve :

Let E local field, res field $\mathbb{F}_q$, unif π $\begin{cases} [E:\mathbb{Q}_p] < \infty \\ \mathbb{F}_q((\pi)). \end{cases}$

Let (R, R^+) $/\mathbb{F}_q$ Tate pair.

Would like to make sense of "$\mathrm{Spa}(E, \mathcal{O}_E) \times_{\mathrm{Spa}\,\mathbb{F}_q} \overset{\overset{S}{\shortparallel}}{\mathrm{Spa}(R, R^+)}$".

Can do it if $\mathrm{char}(E) = p$.

Then $\mathrm{Spa}\,\mathcal{O}_E \times_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}(R^+) = \mathrm{Spa}(\underbrace{R^+[\![\pi]\!]}_{(\pi, [\varpi])\text{-adic top.}})$.

$$\text{so } \mathrm{Spa}\,E \times_{\mathrm{Spa}\,\mathbb{F}_q} S = \mathrm{Spa}(R^+[\![\pi]\!]) \setminus V(\pi [\varpi]).$$
$$= \mathring{\mathbb{D}}^*_S.$$

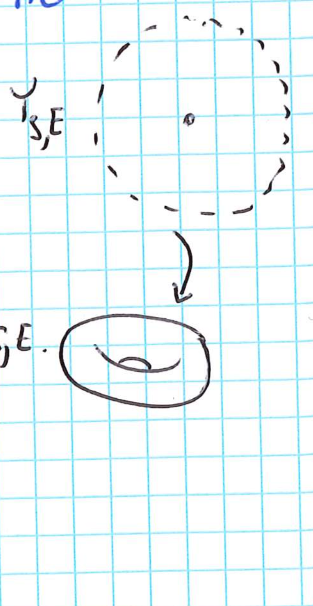
In general, observe that if A is a perfect $\mathbb{F}_q$-alg,

$\exists!$ lift $\tilde{A}/\mathcal{O}_E$ flat π-adically complete
(up to isom.) & $\tilde{A}/\pi = A$

One choice is $\tilde{A} = W(A) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$.

Set : $Y_{S,E} := \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus V(\pi [\varpi])$.

The Frobenius φ_S acts properly discontinuously on $Y_{S,E}$, can define
$$X_{S,E} = Y_{S,E} / \varphi_S.$$



It is an adic space of Krull dim 1 over $\mathrm{Spa}(E)$

"p-adic compact Riemann surface which behaves (arithmetically) like $\mathrm{Spa}(E)$".

Summary (!) of the theory of perfectoid spaces.

Def: • A perfectoid (Tate) ring is a complete $\overset{\text{Tate}}{\text{ring}}$ R, s.t. $\exists\, \varpi$ p.u. with $\varpi^p \mid p$ in R°, R° ϖ-adic ($\Leftrightarrow$ R° ring of def) and $\phi : R^\circ/p \to R^\circ/p$ surjective.

- A perfectoid space is an adic space covered by $\mathrm{Spa}(R, R^+)$, (R, R^+) Tate ring with R perfectoid.

Ex: $R = \mathbb{C}_p$, $\mathbb{F}_q((t^{1/p^\infty}))$, $\mathbb{C}_p\langle T^{1/p^\infty}\rangle$

If $\underset{\text{Tate}}{A}/\mathbb{F}_p$, A perfectoid $\Leftrightarrow$ A perfect.

Tilting construction: R perfectoid ring. Let

$$R^\flat = \varprojlim_{x \mapsto x^p} R \quad \text{(with suitable addition)}$$

right adjoint to Witt vectors. perfectoid ring in char p.

Ex: $K\langle T^{1/p^\infty}\rangle^\flat \simeq K^\flat\langle T^{1/p^\infty}\rangle$.

Th (Scholze): 1) Let (R, R^+) be a perfectoid pair, tilt $(R^\flat, R^{\flat +})$.

The map
$$x \in X = \mathrm{Spa}(R, R^+) \longmapsto x^\flat \in X^\flat := \mathrm{Spa}(R^\flat, R^{\flat +})$$
$$: f \mapsto |f^\#(x)|.$$

induces an homeomorphism $|X| \simeq |X^\flat|$ identifying rational subsets on both sides, so that if $U \subseteq X$ rat. open subset with image $U^\flat \subseteq X^\flat$,

$$(\mathcal{O}_X(U^\flat), \mathcal{O}_X^+(U^\flat)) = (\mathcal{O}_X(U)^\flat, \mathcal{O}_X^+(U)^\flat).$$

$\rightsquigarrow$ implies $\mathcal{O}_X$ sheaf and allows to glue.

Rk: Even better: $X_{\text{ét}} \simeq X^\flat_{\text{ét}}$.

2) X perf'd space. Tilting induces an equivalence:

$$\{\text{perf'd spaces}/X\} \simeq \{\text{perf'd spaces over } X^\flat\}.$$

The Fargues-Fontaine curve & untilts:

$S = \mathrm{Spa}(R, R^+)$, R perfect Tate ring $/\mathbb{F}_q$

Let $T = \mathrm{Spa}(R', R'^+)$ be a perf'd space over E.

$$\text{Maps } T \to Y_{S,E} \Leftrightarrow \underbrace{W_{\mathcal{O}_E}(R^+) \to R'^+}_{\text{s.t. } [\varpi], \pi \text{ map to units of } R'} \Leftrightarrow \underbrace{R^+ \to (R'^+)^\flat}_{\text{adjunction}} \to \text{unit of } R'^\flat$$

$\Leftrightarrow \text{Maps } T^\flat \to S^\flat$.

automatic as T lies $/E$.

In other words, if S perf'd space $/\mathbb{F}_q$,

$$\{\text{untilts } S^\# \text{ of } S \text{ over } E\} \longleftrightarrow S^\# \to Y_{S,E}$$

Moreover, $\Theta : W_{\mathcal{O}_E}(R^+) \to (R^+)^\#$ kernel gen by one elt & non-zero div.

so $S^\# \hookrightarrow Y_{S,E}$ "Cartier divisor" on $Y_{S,E}$ of deg 1.

Classification of vector bundles on the FF-curve.

Take $S = \mathrm{Spa}(C, C^\circ)$, C complete alg. closed $\supset \mathbb{F}_q$.

Recall, An isocrystal is a pair (V, ϕ_V),

V finite dim'l $\breve{E} = W_{\mathcal{O}_E}(\overline{\mathbb{F}}_q)[\frac{1}{\pi}]$ - v.s.

$+ \ \phi_V : V \xrightarrow{\sim} V$ $\phi_{\breve{E}}$-linear aut.

form an E-linear $\otimes$ cat. Isoc_E.

Dieudonné - Manin: Isoc_E is semi-simple

and $\mathrm{Isoc}_E = \bigoplus_{\lambda \in \mathbb{Q}} \mathrm{Isoc}_E^\lambda$

semi-simple obj. of slope λ

$\simeq$ (f.d. E-vs) $\otimes V_\lambda$

$\mathrm{End}(D_\lambda) = D_\lambda$

central simple alg of inv λ -

if $\lambda = \frac{d}{r}$, $(d,r) = 1$

$$\left(V_\lambda = (\breve{E}^r, \ \phi_{V_\lambda} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \pi^d & & & 0 \end{pmatrix} \right)$$

$\overline{\mathbb{F}}_q \subseteq C$, so

$$\begin{tikzcd} Y_{C,E} \arrow[r] \arrow[loop below, "\phi_C"] & \mathrm{Spa}\,\breve{E} \arrow[loop below, "\phi_{\breve{E}}"] \end{tikzcd}$$

Pullback: $\mathrm{Isoc}_E \longrightarrow \{\phi_C\text{-eq vb on } Y_{C,E}\} \simeq VB(X_{C,E})$.

$V \longmapsto \mathcal{E}(V)$.

If $\lambda \in \mathbb{Q}$, write $\cancel{\mathcal{E}}\ \mathcal{O}_{X_{C,E}}(\lambda) = \mathcal{E}(V_{-\lambda})$.

Th (Fargues-Fontaine). The functor $\mathcal{E}(-)$ is essentially surjective. More precisely, any $\mathcal{E} \in VB(X_{C,E})$ admits a HN-filtration $(\mathcal{E}^{\geqslant\lambda})_{\lambda\in\mathbb{Q}}$, s.t. $\mathcal{E}^{\lambda} = gr^{\lambda}\mathcal{E}$ is ss of slope λ

~~and~~ $\mathrm{Isoc}_E^{\lambda} \simeq VB(X_{C,E})^{\lambda}$, and HN-filtration (non-canonically) splits.

Rk: The functor is far from being full!

E.g. $\mathrm{Hom}_E(\mathcal{O}, \mathcal{O}(1))$ is huge: infinite dim'l $/E$.

In fact, interesting relation with LT theory: Let G formal gp law $/\mathcal{O}_E$, let $\tilde{G} = \varprojlim_{\times\pi} G$. Let $S = \mathrm{Spa}(R, R^+)$, $S^\# = \mathrm{Spa}(R^\#, R^{\#+})$ untilt over E. Then

$$\tilde{G}(R^{\#+}) \cong R^{\circ\circ} \longrightarrow H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(1)), \quad x \longmapsto \sum_{i\in\mathbb{Z}} \pi^i [x^{q^{-i}}]$$
$$= H^0(Y_{S,E}, \mathcal{O})^{\varphi=\pi}$$

is an isomorphism

and $H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(1)) \longrightarrow H^0(S^\#, \mathcal{O}_{S^\#}) \longleftrightarrow \log_G : \tilde{G}(R^{\#+}) \to G(R^{\#+}) \to R^{\#+}$.

Rk: Let $\lambda \in \mathbb{Q}$, One can explicitly describe $H^i(X_{C,E}, \mathcal{O}(\lambda))$.

$\begin{cases} \lambda < 0 \Rightarrow \text{always } 0 \text{ unless } i = 1 \\ \lambda \geqslant 0 \Rightarrow \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ } i = 0 \end{cases}$. (More on this next time!).

Rk: Claim: $\pi_1(X_{C,E}) \simeq \mathrm{Gal}(\bar{E}/E)$.

Need to show that $A \mapsto \mathcal{O}_{X_{C,E}} \otimes_E A$ is an equivalence from FET_E and finite étale $\mathcal{O}_{X_{C,E}}$-alg.

Let $\mathcal{E}$ fin. étale $\mathcal{O}_{X_{C,E}}$-alg. Seen as a vb,

$\mathcal{E} = \bigoplus_{i=1}^{r} \mathcal{O}_{X_{C,E}}(\lambda_i)$. Étaleness $\Rightarrow$ perfect trace pairing (self-dual)

$\Rightarrow \sum \lambda_i = 0$.

Let $\lambda = \max \lambda_i$. Assume $\lambda > 0$.

$\mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$ $\curvearrowright$ global section of $\mathcal{E} \otimes \mathcal{O}_{X_{C,E}}(-2\lambda)$.) neg. slopes

$\overset{\cup}{\mathcal{O}_{X_{C,E}}(\lambda) \otimes \mathcal{O}_{X_{C,E}}(\lambda)}$. $\Rightarrow f^2 = 0 \underset{\text{reduced}}{\Longrightarrow} f = 0 \Rightarrow \lambda < 0$ Contradiction.

$\forall f \in H^0(X_{C,E}, \mathcal{O}_{X_{C,E}}(\lambda))$

Vector bundles in families

Def : The v-site is the Grothendieck topology on $\mathrm{Perf}_{\mathbb{F}_q}$ for which a collection

$\{f_i : \underset{S_i}{\cancel{X}_i} \to \underset{S}{\cancel{X}}\}$ of morphisms is a covering

if for each $U \subseteq S$ qc open, $\exists$ finite subset $J \subset I$ ~~qc~~

and $\forall j \in J$, $U_i \subseteq S_i$ qc open

s.t. $U = \bigcup_{i \in J} f_i(U_i)$

"analogue of fpqc top."

Counter example : $\{*\} \sqcup \mathring{\mathbb{D}}_K^*$ not a cover of $\mathbb{D}_K$.

Proposition : The presheaves $\mathcal{O}^+, \mathcal{O}$ are v-sheaves.

Moreover, the v-site is subcanonical.

Rk : $f : T \to S$ analytic adic spaces

Then $|f| : |\underset{T}{\cancel{X}}| \to |\underset{S}{\cancel{X}}|$ is generalizing.

(indeed if $f(x) = y$,

$$\begin{tikzcd} \mathrm{Spa}(k(x), k(x)^+) \arrow[r] \arrow[d] & T \arrow[d, "f"] \\ \mathrm{Spa}(k(y), k(y)^+) \arrow[r] & S \end{tikzcd}$$

and $|\mathrm{Spa}(k(x), k(x)^+)| = \mathrm{Spec}(k(x)^+/\varpi)$)

totally ordered chain of points.

$\Rightarrow$ if f surjective, $|f|$ is a quotient map.

Th : The prestack $\mathrm{Bun} : S \in \mathrm{Perf}_{\mathbb{F}_q} \longmapsto$ gpd of vb on $X_{S,E}$ is a v-stack.

Even better : $\forall a \leq b$, $S \in \mathrm{Perf}_{\mathbb{F}_q} \longmapsto$ gpd $\mathrm{Perf}^{[a,b]}(X_{S,E})$ is a v-stack.