Vector bundles on the Fargues-Fontaine curve and Banach-Colmez spaces

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Recall: E local field, residue field $\mathbb{F}_q$
uniformizer π.

$\left([E:\mathbb{Q}_p] < \infty, \text{ or } E \simeq k((\pi)) \right)$.

$S \in \mathrm{Perf}_{\mathbb{F}_q}$, perfectoid space $/\mathbb{F}_q$.

If $S = \mathrm{Spa}(R, R^+)$, define

$$Y_{S,E} = \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus V(\pi [\varpi])$$

(ϖ = p.u. of R)

$\circlearrowleft \phi$ Set: $X_{S,E} = Y_{S,E}/\phi^{\mathbb{Z}}$.

This construction glues and gives rise to adic spaces $Y_{S,E}$, $X_{S,E}$ over $\mathrm{Spa}\, E$ for any $S \in \mathrm{Perf}_k$.

{In first approximation, can think to $X_{S,E}$ as a family of FF curves (as originally defined) indexed by the points of S.} Will see these are interestingly $\neq$ though.

Recall that the following objects are in natural bijection:
- sections of $Y_S^{\diamond} \to S$
- untilts $S^{\#}$ over E.

Any such untilt $S^\#$ gives rise to a closed Cartier divisor on Y_S and the composite $S^\# \to Y_S \to X_S$ still is one.

Let $\mathrm{Div}^1 = \mathrm{Spd}\, E/\phi^{\mathbb{Z}}$ be the moduli space of closed Cartier divisors of deg 1, i.e. arising from an untilt of S.

The map $\mathrm{Div}^1 \to *$ is proper, rep in spatial $\diamond$, and coh smooth. Note that for $S \in \mathrm{Perf}_{\mathbb{F}_q}$,

$$\mathrm{Div}^1_S := \mathrm{Div}^1 \times S = (\mathrm{Spd}\, E \times S)/\phi_E^{\mathbb{Z}} \times \mathrm{id}$$

$$\neq \quad X_S^\diamond = \text{———} \,/\, \mathrm{id} \times \phi_S^{\mathbb{Z}}.$$

But: $\phi_E \times \phi_S$ acts as id on $|-|$.

$$|X_S| = |X_S^\diamond| = \left|\frac{\mathrm{Spd}\, E \times S}{\mathrm{id} \times \phi_S^{\mathbb{Z}}}\right| \overset{!}{=} \left|\frac{\mathrm{Spd}\, E \times S}{\phi_E^{\mathbb{Z}} \times \mathrm{id}}\right| = |\mathrm{Div}^1_S|$$

Hence have a map $|X_S| \to |S|$ open + closed.

<u>Rk</u>: Div^1 not quasi-separated (although the map $\mathrm{Div}^1 \to *$ is. But $*$ is not!)

In these lectures, want to understand vector bundles on Fargues–Fontaine curve(s).

Key construction: $k = \overline{\mathbb{F}}_q$.

$\breve{E} = W_{O_E}(k)[\frac{1}{p}]$ completion of the maximal unramified ext. of E.
$\circlearrowleft \sigma$

Have a $\otimes$-exact functor, for $S \in \mathrm{Perf}_k$,

$$\mathrm{Isoc}_k \longrightarrow \mathrm{Bun}(X_{S,E})$$

functorially in S

isocrystals over k
$=$ f.d. $\breve{E}$-vs + σ-linear automorphism φ

$$(D,\varphi) \longmapsto \mathcal{E}(D,\varphi) = \text{descent of trivial vb } D \otimes_{\breve{E}} \mathcal{O}_{Y_S} \text{ via } \varphi \otimes \varphi.$$

The category Isoc_k is abelian, semisimple, with simple objects $(D_\lambda, \varphi_\lambda) = \left(\breve{E}^r, \begin{pmatrix} 0 & & \pi^d \\ 1 & \ddots & \\ & 1 & 0 \end{pmatrix}\right)$, $\lambda \in \mathbb{Q}$.
$\lambda = \frac{d}{r}$, $(d,r) = 1$

Denote: $\mathcal{O}(\lambda) = \mathcal{E}((D_{-\lambda}, \varphi_{-\lambda}))$.

Cohomology of vector bundles

For $S = \mathrm{Spa}(K, K^+)$, Y_S is Stein:

$$Y_{S,E} = \bigcup_{I \subset (0,\infty)} Y_{S,E,I}$$

compact interval with rational ends

affinoid sousperfectoid
transition maps have dense images on functions

$\Rightarrow$ If $\mathcal{E}$ v.b. on $X_{S,E}$,

$$R\Gamma(X_{S,E}, \mathcal{E}) \simeq \left(H^0(Y_{S,E}, \mathcal{E}) \xrightarrow{\varphi - id} H^0(Y_{S,E}, \mathcal{E}) \right)$$

Same formula shows that the functor

$$T \in \mathrm{Perf}_S \longmapsto R\Gamma(X_{T,E}, \mathcal{E}|_{X_{T,E}})$$ is

a v-sheaf (reduces to check that if $T' \to T$

v-cover, $0 \to \mathcal{O}(Y_{T,I}) \to \mathcal{O}(Y_{T',I}) \to \mathcal{O}(Y_{T' \times_T T', I}) \to \dots$

is exact. Can be checked after $\hat{\otimes}_E E_\infty \; (= \widehat{E(\pi^{1/p^\infty})})$

In this case, everything becomes perfectoid, and

$Y_{T', E, I} \times_E E_\infty \to Y_{T, E, I} \times_E E_\infty$ is a v-cover

of aff'd perf'd spaces.)

For the vector bundles attached to isocrystals, can make this more explicit.

First we discuss the case of $\mathcal{O}(1)$.

Let G LT formal group law over $\mathcal{O}_E$. Can choose a coordinate $G \simeq \mathrm{Spf}\,\mathcal{O}_E[[X]]$ so that

$$\log_G : G_E \to \mathbb{G}_{a,E}, \quad X \mapsto X + \frac{1}{\pi}X^q + \dots + \frac{1}{\pi^n}X^{q^n} + \dots$$

Define a map of rigid-analytic spaces

$$\log_G : G_E^{ad} \simeq \mathbb{D}_E \to \mathbb{G}_{a,E}^{ad}$$

$S = \mathrm{Spa}(R, R^+)$ aff'd perf'd space $/\mathbb{F}_q$, $S^\#$ untilt over E.

Then $X \mapsto \sum_{i \in \mathbb{Z}} \pi^i [X^{q^{-i}}]$ induces an isom

$$\tilde{G}(R^{\#+}) \xrightarrow{\sim} H^0(X_S, \mathcal{O}(1))$$

$\tilde{G}(R^{\#+})$ $\left(\simeq \lim_{X \mapsto X^p} R^{\#\circ\circ} \simeq R^{\circ\circ}\right)$ s.t. the map "evaluation at $S^\#$"

$$H^0(X_S, \mathcal{O}(1)) \to H^0(S^\#, \mathcal{O})$$ is identified

with $\log_G : \tilde{G}(R^{\#+}) \to G(R^\#) \to R^\#$. Kernel

For any perf'd space $S^\#$ lying over E_∞ with tilt S, get a non-zero section $\mathcal{O} \to \mathcal{O}(1)$ on X_S (indeed, over E_∞, $V_\pi G$ trivialized)

and the seq $0 \to \mathcal{O} \to \mathcal{O}(1) \to \mathcal{O}_{S^\#} \to 0$ is exact.

The previous discussion gives us a corollary:

<u>Cor</u>: There is a well-defined map

$$\begin{aligned} BC(\mathcal{O}(1))\setminus\{0\} &\longrightarrow \mathrm{Div}^1 \\ f &\longmapsto V(f) \end{aligned}$$

which descends to an isom

$$BC(\mathcal{O}(1))\setminus\{0\} / \underline{E}^\times \xrightarrow{\sim} \mathrm{Div}^1 !$$

Pf: By the above, $BC(\mathcal{O}(1)) \simeq \mathrm{Spd}\, \mathcal{O}_{E_\infty}$ so

$$BC(\mathcal{O}(1))\setminus\{0\} \simeq \mathrm{Spd}\, E_\infty$$

in such a way that the map described is well-def'd and corresponds to $\mathrm{Spd}\, E_\infty \xrightarrow[\underline{\mathcal{O}_E^\times}]{} \mathrm{Spd}\, E \longrightarrow \mathrm{Spd}\, E/\varphi^{\mathbb{Z}}$.

quotient by φ $= \pi$ on $BC(\mathcal{O}(1))$.

Hence, closed Cartier divisors of deg 1

$$\longleftrightarrow \{(\mathcal{L}, u)\}/\sim$$

deg 1 line bundle

fiberwise non-zero section on S.

<u>Rk</u>: In particular, $BC(\mathcal{O}(1))\setminus\{0\}$ is a diamond as Div^1 is one and it is perfectoid over it. But $BC(\mathcal{O}(1))$ is only an absolute diamond!

Exo: Make this isomorphism explicit on generic points, when $E = \mathbb{Q}_p$.

$$\underline{\mathcal{M}_C \backslash \mathrm{Sot}} / \mathbb{Q}_p^\times \simeq \{\text{primitive elts of deg } 1\} \qquad \varepsilon \longmapsto \text{ideal generated by } \frac{[1+\varepsilon]-1}{[1+\varepsilon^{1/p}]-1} = x_\varepsilon$$

$B_C^{\varphi=\pi}$

$t_\varepsilon = \log([1+\varepsilon])$

$\mathrm{div}(t_\varepsilon) = \sum_{n\in\mathbb{Z}} \varphi^n([x_\varepsilon])$.

Prop: Let $\lambda \in \mathbb{Q}$.

1) If $\lambda > 0$, $H^1(X_S, \mathcal{O}(\lambda)) = 0$ $\forall S \in \mathrm{Perf}_{\mathbb{F}_q}$ affd perfd and the projection from

$$BC(\mathcal{O}(\lambda)) : S \mapsto H^0(X_S, \mathcal{O}(\lambda))$$

to $*$ is representable in locally spatial diamonds, partially proper and coh smooth.

2) If $0 < \lambda \leq [E:\mathbb{Q}_p]$ or any λ if $\mathrm{char}(E) > 0$, $\exists$ isom $BC(\mathcal{O}(\lambda)) \simeq \mathrm{Spd}\, \mathbb{F}_q[[x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]]$ if $\lambda = \frac{d}{r}$, $(d,r)=1$, $d, r > 0$.

3) $\lambda = 0$. The map $\underline{E} \mapsto BC(\mathcal{O})$ $(: S \mapsto H^0(X_S, \mathcal{O}))$ is an isomorphism and $S \mapsto H^1(X_S, \mathcal{O})$ vanishes after proétale sheafification.

4) $\lambda < 0$. Then $H^0(X_S, \mathcal{O}_{X_S}(\lambda)) = 0$ $\forall S \in \mathrm{Perf}_{\mathbb{F}_q}$.

The projection from
$$BC(\mathcal{O}(\lambda)) : S \mapsto H^1(X_S, \mathcal{O}(\lambda))$$
to $*$ is representable in loc. spatial $\diamond$, partially proper and coh. smooth.

Pf (Sketch): Can always assume $\lambda = n \in \mathbb{Z}$.

1) $S = \mathrm{Spa}(R, R^+)$ affd perfd.

Vanishing of H^1 $\iff$ $\mathrm{coker}(\varphi - \pi^n : B_{R,[1,q]} \to B_{R,[1,1]})$
$= 0$.

Explicit computation.

For the other claim, use $\forall n \in \mathbb{Z}$
$$0 \to \mathcal{O}(n) \to \mathcal{O}(n+1) \to \mathcal{O}_{S^\#} \to 0$$
$$\underset{n \geq 0}{\implies} \quad 0 \to BC(\mathcal{O}(n)) \to BC(\mathcal{O}(n+1)) \to (\mathbb{A}^1_{S^\#})^\diamond \to 0$$

Then induction: $n = 1$ ok
$n > 1$ use this sequence. [ECD 23.1?]

2) Direct computation if $\mathrm{char}(E) > 0$
If $\mathrm{char}(E) = 0$, use Scholze-Weinstein.

3) We use the exact seq

$$0 \to \mathcal{O} \to \mathcal{O}(1) \to \mathcal{O}_{S^\#} \to 0$$

for a given affinoid perfd S with untilt $S^\#$.

Get: $0 \to H^0(X_S, \mathcal{O}) \to H^0(X_S, \mathcal{O}(1)) \xrightarrow{(*)} R^\#$

$\to H^1(X_S, \mathcal{O}) \to 0$.

Know: $(*) = \log_{\tilde{G}}$ surjective pro-étale with kernel $\underline{E}$.

4) $n = -1$ $\quad 0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_{S^\#} \to 0$.

gives $0 \to \underline{E} \to \mathbb{A}^{1\,\diamond}_{S^\#} \to BC(\mathcal{O}(-1)) \to 0$.

(in particular, $H^0(X_S, \mathcal{O}(-1)) = 0$)

+ [ECD] 24.2

For $n < -1$, use $0 \to \mathcal{O}(-n) \to \mathcal{O}(-n+1) \to \mathcal{O}_{S^\#} \to 0$

$\Rightarrow 0 \to \mathbb{A}^{1\,\diamond}_{S^\#} \to BC(\mathcal{O}(-n)) \to BC(\mathcal{O}(-n+1)) \to 0$ (!)

and conclude as in 1).

Note: $BC(\mathcal{O}) = \underline{E}$: Very different from a "classical" complete curve.

Classification over a geometric point

Let $S = \mathrm{Spa}(C)$ completely closed / $\mathbb{F}_q$.

<u>Th</u>: (Fargues-Fontaine) The functor

$$\mathrm{Isoc}_k \to \mathrm{Bun}(X_C)$$

is essentially surjective, i.e. any vb $\mathcal{E}$ on X_C is a direct sum of $\mathcal{O}(\lambda)$, $\lambda \in \mathbb{Q}$.

<u>Rk</u>: Not full at all! E.g. $\mathrm{Hom}(D_0, D_{-1}) = 0$ but $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)) = BC(\mathcal{O}(1))(\mathrm{Spa}\, C)$

$$= B_C^{\varphi = \pi} \text{ inf'dim'l } /E!$$

<u>Pf</u> (Fargues-Scholze) (sketch).

① The case of line bundles.

This requires the schematic curve + ampleness of $\mathcal{O}(1)$ which implies that schematic curve $\setminus$ pt is $\mathrm{Spec}(\mathrm{PID})$

$\Rightarrow$ If $\mathcal{L}$ line bundle on X_C, x classical pt,

$$\mathcal{L}|_{X_C \setminus \{x\}} \simeq \mathcal{O} \text{ i.e. } \mathcal{L} \simeq \mathcal{O}(n[x]).$$

local rings of schematic curve are DVR.

by discussion on $BC(\mathcal{O}(1))$ above,

$$\mathcal{L} \simeq \mathcal{O}(1).$$

So $\mathbb{Z} \to \mathrm{Pic}_X$ is a bijection.
$n \mapsto \mathcal{O}(n)$

(2) If $\mathcal{E}$ is a vb on X_C, define
$$\deg(\mathcal{E}) = \deg(\det \mathcal{E})$$
deduced from isom in (1).

Set $$\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}$$

HN formalism $\Rightarrow$ for any vb $\mathcal{E}$, $\exists!$ exhaustive separating $\mathbb{Q}$-indexed filtration $\mathcal{E}^{\geq\lambda}$ of $\mathcal{E}$ by subbundles s.t. $\forall \lambda \in \mathbb{Q}$ $\mathcal{E}^{\lambda} := \mathcal{E}^{\geq\lambda} / \bigcup_{\lambda' > \lambda} \mathcal{E}^{\geq\lambda'}$ ss. of slope λ.

Compatible with change of C and E.

Start pf of thm: induction on n.

$n=1$, ok.

$n>1$. Assume for $\mathrm{rk} \leq n-1$. If $\mathcal{E}$ not semistable, follow by induction + $H^1(X_C, \mathcal{O}(\lambda)) = 0$ if $\lambda > 0$.

Assume $\mathcal{E}$ ss slope $\lambda = \frac{d}{r}$, $(d,r)=1$, $r>0$.

Assume $\exists \neq 0$ map $\mathcal{O}(\lambda) \to \mathcal{E}$. The category of

ss bundles of slope λ is abelian with simple object the stable ones. Thus such a map is injective and quotient ss of slope λ, hence $\simeq \mathcal{O}(\lambda)^m$ for some m. Since $\mathrm{Ext}^1_{X_C}(\mathcal{O}(\lambda), \mathcal{O}(\lambda)) = 0$, we win.
Up to increasing E, can assume $\lambda \in \mathbb{Z}$ and then $\lambda = 0$ by twisting.

Moreover, we are free to increase C to C'.
Indeed, consider the v-sheaf $S \mapsto \{\text{isom } \mathcal{E} \simeq \mathcal{O}^n\}$.
By Prop, $\underline{\mathrm{Aut}}(\mathcal{O}^n) = \underline{\mathrm{Gl}_n(E)}$, hence it is a v-pseudotorsor under $\underline{\mathrm{Gl}_n(E)}$. If $\exists$ ext C' s.t. $\exists$ non-zero section (and thus can iso), it is a v-torsor.
By v-descent of $\underline{\mathrm{Gl}_n(E)}$-torsors it is rep by pro-étale adic space over $\mathrm{Spa}(C)$, hence has a section.

Let $d \geq 0$ minimal s.t. $\exists$ injective map
$$\mathcal{O}(-d) \to \mathcal{E}.$$
after any extension of C. (exists as $\mathcal{O}(1)$ ample). By minimality of d, $\mathcal{F} = \mathcal{E}/\mathcal{O}(-d)$ v.b.

Want $d=0$, assume $d>0$.

Key case: case $d=1$, $\mathcal{F}$ semistable.
(so $\mathcal{F} \simeq \mathcal{O}(\frac{1}{n+1})$.)

Hence, reduced to:

(3) Prop: $0 \to \mathcal{O}(-1) \to \mathcal{E} \to \mathcal{O}(\frac{1}{n}) \to 0$
extension of vb on X_C.
Then $\exists\, C' | C$ s.t. $H^0(X_{C'}, \mathcal{E}) \neq 0$.

Pf: Assume not. Then get an injection of v-sheaves $f: \mathcal{BC}_C(\mathcal{O}(\frac{1}{n})) \to \mathcal{BC}_C(\mathcal{O}(-1)) = \mathbb{A}^{1,\diamond}_{C^\#}/\underline{E}$.
(injectivity can be checked on field-valued pts, as both v-sheaves are locally spatial).

Image cannot be contained in classical pts (i.e. the ones coming from classical points of $\mathbb{A}^1_{C^\#}$), since they form a totally disconnected set. Enlarge C: must be type II.

Thus, image $\supseteq$ nonempty open subset of $\mathcal{BC}(\mathcal{O}(-1))$ after <u>enlarging C</u>.

To see that, note the following lemma:

$K = \mathbb{C}^{\flat}$

Lemma: $x \in \mathbb{A}^1_K(K)$, $\rho > 0$, x_ρ be the Gauss norm of radius ρ centred at x. The preimage of x_ρ in $\mathbb{A}^1_{K(x_\rho)}$ contains open disk with radius ρ around x.

Pf: Can assume $x = 0$, $\rho \leq 1$.

$$\rho : K\langle T\rangle \to K(x_\rho).$$
$$T \mapsto t$$

$y \in \mathbb{D}_{K(x_\rho)}$, corresponding to $K(x_\rho)\langle T\rangle \to K(x_\rho)(y)$.
$$T \mapsto u.$$

Assume $y \in$ open disk of radius ρ at x_ρ.

Then $|u - t| < \rho = |t|$. $\Rightarrow \forall n \geq 1$, $|u^n - t^n| < \rho^n$.

If $f = \sum a_n T^n \in K\langle T\rangle$,
$$\left|\sum a_n(u^n - t^n)\right| \leq \sup |a_n| \rho^n = \sum a_n t^n.$$
$$\text{so } \left|\sum a_n u^n\right| = \left|\sum a_n t^n\right| = |f(x_\rho)|. \quad \blacksquare$$

By translating this $\neq \emptyset$ open subset, can assume contains origin and then by contracting action of $E^\times$, image must be everything. So
$$BC(\mathcal{O}(\tfrac{1}{n})) \simeq \mathbb{A}^{1,\diamond}/\underline{E} \Rightarrow \text{RHS perf'd space.}$$

Absurd if $\mathrm{chr}(E) = 0$.

In general:

Pick a $\neq 0$ map $BC(\mathcal{O}(\frac{1}{n})) \to \mathbb{A}^{1\,\diamond}_{C^\#}$

If f iso, f^{-1} followed by this map gives

$$\mathbb{A}^{1\,\diamond}_{C^\#}/\underline{E} \to \mathbb{A}^{1\,\diamond}_{C^\#}. \quad \neq 0.$$

Maps $\mathbb{A}^{1\,\diamond}_{C^\#} \to \mathbb{A}^{1\,\diamond}_{C^\#}$: convergent power series

$g(X)$ st $g(X+Y) = g(X)+g(Y)$, $g(\lambda X) = \lambda g(X)$ $\forall \lambda \in E$.

$g(\pi X) = \pi g(X) \Rightarrow g(X) = \alpha X$, $\alpha \in C^\#$.

If $\alpha \neq 0$, iso, but required to have kernel $\underline{E}$... ∎

Application: X_C is geometrically simply connected.

Pf: Need to see that pullback induces an equivalence

$\{$finite étale E-alg$\} \simeq \{$finite étale $\mathcal{O}_{X_C}$-alg$\}$.

$\mathcal{E}$ finite étale $\mathcal{O}_{X_C}$-alg. Classification: $\mathcal{E} = \bigoplus \mathcal{O}(\lambda_i)$.

Non degenerate trace pairing $\Rightarrow \sum \lambda_i = 0$. Let $\lambda = \max(\lambda_i)$.

Multiplication restricts $\mathcal{O}(\lambda) \otimes \mathcal{O}(\lambda) \to \mathcal{E} \Rightarrow \lambda \leq 0$. Thus $\lambda_i = 0$ $\forall i$. Hence $\mathcal{E} = A \otimes_E \mathcal{O}_{X_C}$, A finite étale E-alg. ∎

Relation with p-divisible groups: ($E=\mathbb{Q}_p$)

H/k p-divisible group. $D = D(H)$ Dieudonné module / $\breve{\mathbb{Q}}_p$

$\mathcal{E} = \mathcal{E}(D)\otimes\mathcal{O}(1)$. $C^\#$ untilt, G p-div gp / $\mathcal{O}_{C^\#}$ lifting $H\otimes_k \mathcal{O}_{C^\#}/p$.

$i: \mathrm{Spa}(C^\#) \hookrightarrow X_C$.

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[rr, bend right, "f_G"'] & i_* i^* \mathcal{E} \cong i_*(D\otimes_{\breve{\mathbb{Q}}_p} C^\#) \arrow[r, two heads] & i_*(\mathrm{Lie}\, G \otimes_{\mathcal{O}_{C^\#}} C^\#) \end{tikzcd}$$

$\mathcal{E}(G) = \ker(f_G)$.

p-adic comparison thm $\Rightarrow$ $\mathcal{E}(G) \cong V_p(G)\otimes_{\mathbb{Q}_p} \mathcal{O}_{X_C}$

In particular, all modifications of $\mathcal{E}$ arising from p-div gps as above are trivial. Was used by FF in combination with description of image of period map (sending G to $(D\otimes_{\breve{\mathbb{Q}}_p} C^\# \to \mathrm{Lie}\, G\otimes_{\mathcal{O}_{C^\#}} C^\#)$) to prove classification theorem.

Punctured BC spaces

Rk: Also a version for H^1 if slope < 0.

Prop: $S \in \mathrm{Perf}_{\mathbb{F}_q}$. $\mathcal{E}$ vb on X_S. Then

$$BC(\mathcal{E}) : T \mapsto H^0(X_T, \mathcal{E}|_{X_T})$$

is a locally spatial & partially proper $\diamond$ over S.

Moreover, $\dfrac{BC(\mathcal{E})\setminus 0}{E^\times}$ is locally spatial diamond, proper over S.

Pf: $\mathcal{O}(1)$ ample $\Rightarrow \exists\ \mathcal{O}(-n)^m \twoheadrightarrow \mathcal{E}^\vee$.

Dualizing, $0 \to \mathcal{E} \to \mathcal{O}(n)^m \to \mathcal{F} \to 0$

$\mathcal{F}$ vb.

$BC(\mathcal{F})$ is separated (enough zero section is closed, but being zero can be checked at every untilt, there defines Zariski closed conditions)

$\Downarrow$

$BC(\mathcal{E}) \subset BC(\mathcal{O}(n))^m$ closed

For the second, assume S qcqs. Enough to prove for $(BC(\mathcal{E})\setminus 0)/\pi^{\mathbb{Z}}$. Deduced from the fact that...

if T locally spectral looking like this space, γ automorphism s.t. $T_0 = \mathrm{Fix}(\gamma) \subset T$ spectral. Assume action γ^n diverges (resp. converges to) T_0 for $n \to -\infty$, resp. $n \to \infty$. Then $(T\setminus T_0)/\gamma^{\mathbb{Z}}$ spectral.

An application:

Th (Kedlaya-Liu) $S \in \mathrm{Perf}_{\mathbb{F}_q}$, $\mathcal{E}$ vb on X_S rank n.

Then: 1) $HN(\mathcal{E}) : (s : \mathrm{Spa}\, C \to S) \mapsto HN(\mathcal{E}_{|X_s})$ is upper semicontinuous.

2) If $HN(\mathcal{E})$ constant, $\exists$ global HN filtration which moreover splits pro-étale locally on S.

Pf: 1) $\forall s$, $HN(\mathcal{E}_s)$ = convex hull of (i, d_i)

d_i = max integer s.t. $H^0(X_C, \Lambda^i \mathcal{E}_s(-d_i)) \neq 0$.

Thus enough:

$\{s,\ H^0(X_s, \mathcal{F}_{|X_s}) \neq 0\}$ closed in S if $\mathcal{F}$ v.b. on X_S.

but this image $\Big(\underbrace{\dfrac{\mathcal{BC}(\mathcal{F}) \setminus 0}{\underline{E}^\times}}_{\text{proper}} \to S \Big)$.

For 2) enough to prove that v-locally on S, $\mathcal{E} \simeq \bigoplus \mathcal{O}(\lambda_i)$. Indeed, then the HN filtration exists locally and thus descends by unicity.

Induction on $\mathrm{rank}(\mathcal{E})$. (then trivializing each $\mathcal{E}^\lambda$ = pro-étale torsor)

λ = maximal slope of $\mathcal{E}$.

<u>Claim</u>: v-locally on S, $\exists\ \mathcal{O}(\lambda) \to \mathcal{E}$ nonzero in each fiber.

Equivalently need to find $\mathcal{O} \to \mathcal{F} = \underline{\mathrm{Hom}}(\mathcal{O}(\lambda), \mathcal{E})$ non zero fiberwise.

But $\mathcal{BC}(\mathcal{F}) \setminus 0 \to \mathcal{BC}(\mathcal{F}) \setminus 0 / \underline{E}^\times \to S$

is a v-cover s.t. such a map exists.

Indeed: 1^{st} map $= \underline{E}^\times$-torsor, hence v-cover.

2^{nd} map $=$ proper $+$ surjective on geom pts

$\Longrightarrow$ surjective.

(grl length: [ECD] 2.11)

Now, dual map $\mathcal{E}^\vee \to \mathcal{O}(-\lambda)$ surjective

(can be checked on generic pts, use $\mathcal{O}(\lambda)$ stable)

So $\mathcal{E}' = \mathrm{coker}(\mathcal{O}(\lambda) \to \mathcal{E})$ v.b.

By additivity of $HN(-)$, $HN(\mathcal{E}')$ also constant, so can apply inductive hypothesis. After a further pro-étale cover, can split the ext. $\blacksquare$

Fargues-Scholze also prove an absolute result:

<u>Th</u>: If D is an isocrystal with either only negative slopes or only positive slopes, then
$BC(D)\setminus\{0\}$ is a spatial $\diamond$ and the quotient

$$\underline{BC(D)\setminus\{0\}}_{\underline{E}^\times} \to * \text{ is proper, rep in spatial } \diamond \text{ and coh smooth.}$$

The proof of second part follows from what has been done already, but first part is hard if slopes > 0.
(i.e. vb with slopes < 0)

<u>Ex</u>: $\kappa d \leq [E:\mathbb{Q}_p]$,

$$BC(\mathcal{O}(d))\setminus\{0\} = \mathrm{Spd}\left(k[[x_1^{1/p^\infty},\dots,x_d^{1/p^\infty}]]\right)\setminus V(x_1,\dots,x_d).$$

qcqs perf'd space.

But not qcqs after base change to $\mathrm{Spd}\, C$!
(same for simply connected).

$$BC(\mathcal{O}(-1))\setminus\{0\} = \frac{BC(\mathcal{O}(\tfrac{1}{2}))\setminus\{0\}}{\ker(\mathrm{Nrd}: D_{1/2}^\times \to E^\times)}.$$