Banach-Colmez spaces
Arthur-César Le Bras

Banach-Colmez spaces were introduced by Colmez in [1] (under the name "Espaces de Banach de Dimension Finie") almost fifteen years ago to give a new proof the conjecture "weakly admissible implies admissible" in $p$-adic Hodge theory. The goal of the talk was to show why they are important and ubiquitous.

Let $C$ be the completion of an algebraic closure of $\mathbb{Q}_p$. Let $\mathrm{Perf}_C$ be the category of perfectoid spaces over $C$, and $\mathrm{Perf}_{C,\mathrm{proet}}$ be the big pro-étale site of $C$ (the above category endowed with the pro-étale topology). We will look at presheaves on the category $\mathrm{Perf}_C$ with values in the category of $\mathbb{Q}_p$-topological vector spaces, which are sheaves on $\mathrm{Perf}_{C,\mathrm{proet}}$ when viewed simply as presheaves of $\mathbb{Q}_p$-vector spaces. Such a functor $F$ is called a Banach sheaf when $F(X)$ is a Banach space for all affinoid perfectoid $X$. Here are two simple examples of Banach sheaves: the constant sheaf $V$, for any finite dimensional $\mathbb{Q}_p$-vector space $V$; the sheaf $W \otimes C \mathcal{O}$, for any finite dimensional $C$-vector space $W$.

The following definition looks a bit different from Colmez’s one, but is actually equivalent.

**Definition 0.1.** An effective Banach-Colmez space is a Banach sheaf $F'$ which is an extension

$$0 \to V \to F' \to W \otimes C \mathcal{O} \to 0,$$

$V$ (resp. $W$) being a finite dimensional $\mathbb{Q}_p$-vector space (resp. a finite dimensional $C$-vector space). A Banach-Colmez space is a Banach sheaf $F$ which is a quotient

$$0 \to V' \to F' \to F \to 0,$$

where $F'$ is an effective Banach-Colmez space and $V'$ a finite dimensional $\mathbb{Q}_p$-vector space. The category of Banach-Colmez spaces will be denoted $\mathcal{BC}$.

To any presentation of a Banach-Colmez space as in the definition, we associate two integers: its dimension $\dim_C W$ and its height $\dim_{\mathbb{Q}_p} V - \dim_{\mathbb{Q}_p} V'$.

The definition of the category of Banach-Colmez spaces may look a bit strange, but Colmez proved the following difficult theorem ([1]).

**Theorem 0.2.** The category $\mathcal{BC}$ is an abelian category. The functor $F \mapsto F(C)$ is exact and conservative on $\mathcal{BC}$.

Moreover, the functions dimension and height do not depend on the presentation and define two additive functions on $\mathcal{BC}$.

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1. By definition, a sequence of Banach sheaves is said to be exact if it is so as a sequence of sheaves of $\mathbb{Q}_p$-vector spaces on $\mathrm{Perf}_{C,\mathrm{proet}}$. 

At this point it is not clear that there exist interesting examples of Banach-Colmez spaces apart the obvious ones. To construct geometrically such examples, one can use $p$-divisible groups, as was observed by Fargues ([2]). Let $G$ be a $p$-divisible group over $\mathcal{O}_C$. Its universal cover $\tilde{G}$ is the sheaf which associates to any perfectoid algebra $R$ over $C$ the $\mathbb{Q}_p$-vector space $\tilde{G}(R) = \varprojlim \times_{p} \varprojlim_{k} \varprojlim_{n} G[p^n](R)$.

This sheaf is representable by a perfectoid space over $C$. For example if $G = \mathbb{Q}_p / \mathbb{Z}_p$, $\tilde{G} = \mathbb{Q}_p^{2}$. In general, one has an exact sequence of pro-étale sheaves

$$0 \to V(G) \to \tilde{G} \overset{\log}{\to} \text{Lie}(G)[p^{-1}] \otimes \mathcal{O} \to 0,$$

$V(G)$ being the rational Tate module of $G$. As moreover $\tilde{G}(R)$ is a Banach space for any perfectoid $C$-algebra $R$, this exact sequence shows that universal covers of $p$-divisible groups are examples of effective Banach-Colmez spaces! Actually, one can prove the following result

**Theorem 0.3.** Universal covers of $p$-divisible groups over $\mathcal{O}_C$ are Banach-Colmez spaces and any Banach-Colmez space is the quotient of the universal cover of a $p$-divisible group by the Banach-Colmez space $V$ associated to some finite dimensional $\mathbb{Q}_p$-vector space $V$.

This result has two consequences. The first one is the

**Corollary 0.4.** Banach-Colmez spaces are diamonds over $\text{Spa}(C^\flat)$.

The deep results of Fargues [2] and Scholze-Weinstein [4] on $p$-divisible groups allow to describe universal covers in terms of $p$-adic Hodge theory. The second consequence of the theorem is thus that one can get many explicit examples of Banach-Colmez spaces by playing with Fontaine rings. Here is an example: for any $\lambda = d/h \in \mathbb{Q}$, $\lambda \geq 0$, the functor $U_\lambda : R \mapsto B^{+}_{\text{crys}}(R^\flat / p^\lambda)$ is a Banach-Colmez space. For instance, $U_1 = \tilde{\mu}_p^\infty$ and the exact sequence (1) for $G = \mu_\infty$ evaluated on $C$ becomes identified with the famous exact sequence

$$0 \to \mathbb{Q}_p, t \to (B^{+}_{\text{crys}})^{\varphi = p} \overset{\theta}{\to} C \to 0.$$

To completely elucidate the nature and the structure of the category $\mathcal{BC}$, we now turn to the relation with the Fargues-Fontaine curve $X$ (for $E = \mathbb{Q}_p$, $F = C^\flat$).

Let

$$\text{Coh}^{0,-}(X) = \{ \mathcal{F} \in D(X), H^i(\mathcal{F}) = 0 \text{ for } i \neq -1, 0; H^{-1}(\mathcal{F}) < 0; H^0(\mathcal{F}) \geq 0 \},$$

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2. This sheaf is representable by the perfectoid space $\text{Spa}(\mathcal{C}^0(\mathbb{Q}_p, C), \mathcal{C}^0(\mathbb{Q}_p, \mathcal{O}_C))$.

3. Here we implicitly identify $\text{Perf}_{C,\text{proet}}$ with $\text{Perf}_{C^\flat,\text{proet}}$, using Scholze’s equivalence.
where $D(X)$ is the bounded derived category of the abelian category $\text{Coh}(X)$ of coherent sheaves on $X$, and where for $\mathcal{E} \in \text{Coh}(X)$, the notation $\mathcal{E} \geq 0$ (resp. $\mathcal{E} < 0$) means that all the slopes of $\mathcal{E}$ are non negative (resp. negative). This full subcategory of $D(X)$ is actually an abelian category (this is a consequence of the general theory of \textit{tilting} and \textit{torsion pairs}), and is endowed with a degree function $\text{deg}^{0,-}$ and a rank function $\text{rk}^{0,-}$.

For any perfectoid space $S$ over $C^\flat$, there exists a relative version $X_S$ of the curve (for $S = \text{Spa}(C^\flat)$, this is just the usual Fargues-Fontaine curve $X$). Although there is no morphism of adic spaces $X_S \to S$, one has a morphism of sites $\tau$ from $\text{Perf}_{C^\flat, \text{proet}}$ to the big pro-étale site of $X$. In particular, one can associate to any complex of coherent sheaves $\mathcal{F}$ on $X$ a sheaf $R^j\tau_*\mathcal{F}$ on $\text{Perf}_{C^\flat, \text{proet}}$, for any $j \geq 0$.

**Theorem 0.5.** The functor $R^0\tau_*$ induces an equivalence of categories $\text{Coh}^{0,-}(X) \simeq \mathcal{BC}$.

Under this equivalence the functions $\text{deg}^{0,-}$ and $-\text{ht}$ (resp. $\text{rk}^{0,-}$ and $\text{dim}$) correspond to each other.

For example, $R^0\tau_*$ sends $\mathcal{O}_X$ to $\mathbb{Q}_p$, $i_{\infty,*}C$ to $\mathcal{O}$, and $\mathcal{O}_X(-1)[1]$ to $\mathcal{O}/\mathbb{Q}_p$. This result gives a precise meaning to the idea that all Banach-Colmez spaces can be obtained by using $H^0$ and $H^1$ of coherent sheaves on the Fargues-Fontaine curve. It also shows that the category $\mathcal{BC}$ only depends on $C^\flat$.

Using this result and the corollary 0.4, one can show that automorphism groups of vector bundles on $X$ are diamonds : see [3, Prop. 2.5]. For example, $\text{Aut}(\mathcal{O}_X^n) = \text{GL}_n(\mathbb{Q}_p)$ (and not the algebraic group $\text{GL}_n$!). This point is important for Fargues’s conjecture.

**RÉFÉRENCES**


