The work of Drinfeld
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The goal of the talk was to justify the central role played by moduli spaces of shtukas in the Langlands program, by giving a brief overview of the work of Drinfeld on the global Langlands correspondence for function fields. This is a big and deep subject and we decided to focus on the results of [3] and on the relation between elliptic modules and shtukas with two legs. Our discussion follows closely [1] and [2].

As usual, let \( k = \mathbb{F}_q \), \( X \) a smooth projective, geometrically connected curve over \( k \) and \( F = k(X) \). Choose a point \( \infty \in |X| \), and assume for simplicity that \( \text{deg}(\infty) = 1 \).

Let \( F_\infty \) be the completion of \( F \) at \( \infty \), \( C_\infty \) be the completion of a separable closure \( \overline{F}_\infty \) of \( F_\infty \), and \( A = H^0(X \setminus \{\infty\}, \mathcal{O}) \).

1. Elliptic modules

1.1. Definition. The seed of shtukas were Drinfeld’s elliptic modules. Let \( \mathbb{G}_a \) be the additive group, and \( K \) a characteristic \( p \) field. We set \( K\{\tau\} = K \otimes_{\mathbb{Z}} \mathbb{Z}[\tau] \), with multiplication given by

\[
(a \otimes \tau^i)(b \otimes \tau^j) = ab^{i+j} \tau^{i+j}.
\]

We have an isomorphism \( K\{\tau\} \cong \text{End}_K(\mathbb{G}_a) \) sending \( \tau \) to \( X \mapsto X^p \). If \( a_m \) is the largest non-zero coefficient, then the degree of \( \sum_{i=0}^{m} a_i \tau^i \in K\{\tau\} \) is defined to be \( p^m \). The derivative is defined to be the constant term \( a_0 \).

Definition 1.1. Let \( r > 0 \) be an integer and \( K \) a characteristic \( p \) field. An elliptic \( A \)-module of rank \( r \) is a ring homomorphism

\[
\phi: A \to K\{\tau\}
\]

such that for all non-zero \( a \in A \), \( \text{deg} \phi(a) = |a|_\infty^r \).

Let \( S \) be a scheme of characteristic \( p \). An elliptic \( A \)-module of rank \( r \) over \( S \) is a \( \mathbb{G}_a \)-torsor \( \mathcal{L}/S \), with a morphism of rings \( \phi: A \to \text{End}_S(\mathcal{L}) \) such that for all points \( s: \text{Spec} \, K \to S \), the fiber \( \mathcal{L}_s \) is an elliptic \( A \)-module of rank \( r \).

Remark 1.2. The function \( a \mapsto \phi(a)' \) (the latter meaning the derivative of \( \phi(a) \)) defines a morphism of rings \( i: A \to \mathcal{O}_S \), i.e. a morphism \( \theta: S \to \text{Spec} \, A \).

1.2. Level structures and moduli space. Let \( I \) be an ideal of \( A \). Let \( (\mathcal{L}, \phi) \) be an elliptic module over \( S \). Assume for simplicity that \( S \) is an \( A[I^{-1}] \)-scheme, i.e. the map \( \theta \) factors through \( \theta: S \to \text{Spec} \, A[V(I)] \).

Let \( \mathcal{L}_I \) be the group scheme defined by the equations \( \phi(a)(x) = 0 \) for all \( a \in I \). This is an étale group scheme over \( S \) with rank \( \#(A/I)^r \). An \( I \)-level structure on \( (\mathcal{L}, \phi) \) is an \( A \)-linear isomorphism \( \alpha: (I^{-1}/A)S^r \cong \mathcal{L}_I \).

Choose \( 0 \subseteq I \subsetneq A \). We have a functor

\[
F_I^r: A[I^{-1}] - \text{Sch} \to \text{Sets}
\]

sending \( S \) to the set of isomorphism classes of elliptic \( A \)-modules of rank \( r \) with \( I \)-level structure, with \( \theta \) being the structure morphism.

Theorem 1.3 (Drinfeld). \( F_I^r \) is representable by a smooth affine scheme \( \mathcal{M}^r_I \) over \( A[I^{-1}] \).

2. Analytic theory of elliptic modules

2.1. Description in terms of lattices. Let \( \Gamma \) be an \( A \)-lattice in \( C_\infty \) (that is, a discrete additive subgroup of \( C_\infty \) which is an \( A \)-module.) Then we define

\[
e_{\Gamma}(x) = \prod_{\gamma \in \Gamma^{-1}} (1 - x/\gamma).
\]
Drinfeld proved that this is well-defined for all \( x \in \mathbb{C}_\infty \), and induces an isomorphism of abelian groups \( e_\Gamma : \mathbb{C}_\infty / \Gamma \sim \mathbb{C}_\infty \). This allows to define a function \( \phi_\Gamma : A \to \text{End}_{\mathbb{C}_\infty} (G_a) \), by transporting the \( A \)-module structure on the left-hand side to the right-hand side, which only depends on the homothety class of the \( A \)-lattice \( \Gamma \).

The following theorem is reminiscent of the description of elliptic curves over \( \mathbb{C} \).

**Theorem 2.1** (Drinfeld). The function \( \Gamma \mapsto \phi_\Gamma \) induces a bijection between

\[
\left\{ \begin{array}{l}
\text{rank } r \text{ projective } A\text{-lattices} \\
\text{in } \mathbb{C}_\infty / \text{homothety}
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{l}
\text{rank } r \text{ elliptic } A\text{-modules} \\
\text{over } \mathbb{C}_\infty \text{ such that } \phi(a)\Gamma = a
\end{array} \right\}
\]

**Remark 2.2.** Under this bijection, an \( I \)-level structure equivalent to an \( A \)-linear isomorphism \( (A/I)^r \cong \Gamma / \Gamma I \) for the lattices.

### 2.2. Uniformization

We now try to parametrize the objects on the left hand side of \([2.1]\).

Let \( Y \) be a projective \( A \)-module of rank \( r \). Then we have a bijection

\[
\left\{ \begin{array}{l}
\text{homothety classes of } A\text{-lattices in } \mathbb{C}_\infty \\
\text{isomorphic to } Y \text{ as } A\text{-modules}
\end{array} \right\} \leftrightarrow \mathbb{C}_\infty^\times \text{Inj}(F_\infty \otimes_A Y, \mathbb{C}_\infty)/\text{GL}_A(Y).
\]

Next we observe that there is a bijection (after fixing an identification \( F_\infty \otimes_A Y = F_\infty^0 \))

\[
\mathbb{C}_\infty^\times \text{Inj}(F_\infty \otimes_A Y, \mathbb{C}_\infty) \leftrightarrow \mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus \bigcup (F_\infty^\ast \text{-rational hyperplanes}),
\]

given by sending \( u \in \text{Inj}(F_\infty \otimes_A Y, \mathbb{C}_\infty) \) to \([u(e_1) : \ldots : u(e_r)]\) \(((e_1, \ldots, e_r) \text{ is the canonical basis of } F_\infty^0)\). The right-hand side is the set of \( \mathbb{C}_\infty^\ast \)-points of the famous **Drinfeld upper half-space** \( \Omega^r \).

As \( \text{Spec } A = X \setminus \{ \infty \} \), a projective \( A \)-module of rank \( r \) is the same as a vector bundle of rank \( r \) on \( X \setminus \{ \infty \} \). Using Weil’s adèlic description of vector bundles, one finally gets

\[
M_f^r(\mathbb{C}_\infty) \cong \text{GL}_r(F)/\Omega^r(\mathbb{C}_\infty) \times \text{GL}_r(A_\mathbb{Q}^\infty) / \text{GL}_r(\hat{A}, I),
\]

where \( \text{GL}_r(\hat{A}, I) := \ker \left( \text{GL}_r(\hat{A}) \rightarrow \text{GL}_r(A/I) \right) \). This bijection can be upgraded into an isomorphism of rigid analytic spaces:

**Theorem 2.3** (Drinfeld). One has an isomorphism of rigid analytic spaces over \( F_\infty^0 \):

\[
M_f^{r, \text{an}} = \text{GL}_r(F)/\Omega^r \times \text{GL}_r(A_\mathbb{Q}^\infty) / \text{GL}_r(\hat{A}, I).
\]

### 3. Cohomology of \( M_f^2 \) and Global Langlands for \( \text{GL}_2 \)

#### 3.1. Cohomology of the Drinfeld upper half plane

We then briefly outlined Drinfeld’s proof of global Langlands for \( \text{GL}_2 \) using the moduli space of elliptic modules. Set \( r = 2 \), and \( \Omega := \Omega^2 \).

Then one has

\[
\Omega(\mathbb{C}_\infty) = \mathbb{P}^1(\mathbb{C}_\infty) \setminus \mathbb{P}^1(F_\infty).
\]

There is a map \( \lambda \) from \( \Omega(\mathbb{C}_\infty) \) to the Bruhat-Tits tree, sending \((z_0, z_1)\) to the homothety class of the norm on \( F_\infty^2 \) defined by

\[
(a_0, a_1) \in F_\infty^2 \mapsto |a_0z_0 + a_1z_1|,
\]

and one can think to \( \Omega \) as being a tubular neighborhood of the Bruhat-Tits tree. Using \( \lambda \), one gets a quite explicit description of the geometry of the rigid analytic space \( \Omega \) and proves that there is a \( \text{GL}_2(F_\infty^0) \)-equivariant isomorphism:

\[
H^1_{\text{ét}}(\Omega_{\mathbb{C}_\infty}, \mathbb{Q}_l) = (\mathbb{C}_\infty^\ast(\mathbb{P}^1(\mathbb{C}_\infty), \mathbb{Q}_l)/\mathbb{Q}_l)^* \cong \text{St}^*.
\]
3.2. Cohomology of $M_f^2$. Now we use the uniformization of $M_f^2$ (theorem 2.3). Rewriting it as follows:

$$M_f^2, \text{an} = \left( \Omega \times \GL_2(F) \backslash \GL_2(A_F) / \GL_2(\hat{A}, I) \right) / \GL_2(F_{\infty}).$$

and using the Hochschild-Serre spectral sequence, we deduce a $\GL_2(A_F) \times \Gal(F_{\infty}/F_{\infty})$-equivariant isomorphism.

$$H^1_{\et}(M_f^2 \otimes F, \mathbb{Q}) \cong \Hom_{\GL_2(F_{\infty})}(\text{St}, C_0^\infty(\GL_2(F) \backslash \GL_2(A_F) / \GL_2(\hat{A}, I))) \otimes \text{sp},$$

where sp is a 2-dimensional representation of $\Gal(F_{\infty}/F_{\infty})$ corresponding to the Steinberg representation by local Langlands. Drinfeld shows that

$$\lim_{\longrightarrow} H^1_{\et}(M_f^2 \otimes F, \mathbb{Q}) = \bigoplus_{\pi} \pi^\infty \otimes \sigma(\pi)$$

where $\pi$ runs over cuspidal automorphic representations of $\GL_2(A_F)$ with $\pi_\infty \cong \text{St}$. Here $\sigma(\pi)$ is a degree two $\Gal(F/F)$-representation. Moreover, Drinfeld shows that at unramified places, $\pi_v$ and $\sigma(\pi_v)$ correspond to each other by local Langlands.

Remark 3.1. This result is still quite far from the global Langlands correspondence for $GL_2$ over $F$, but it nevertheless allows to construct the local Langlands correspondence for $GL_2$ over $K$, a characteristic $p$ local field, as was explained during the talk, by combining this global construction with the decomposition of global $L$ and $\epsilon$-factors as products of local constants (which is known to hold in positive characteristic) and a trick of twisting by a sufficiently ramified character. See [2].

4. FROM ELLIPTIC MODULES TO SHTUKAS

The relation between elliptic modules and shtukas passes through an intermediate object called an elliptic sheaf.

Definition 4.1. An elliptic sheaf of rank $r > 0$ with pole at $\infty$ is a diagram

$\cdots \rightarrow \mathcal{F}_{i+1} \xrightarrow{j_{i+1}} \mathcal{F}_i \xrightarrow{j_i} \mathcal{F}_{i-1} \xrightarrow{\tau_{i-1}} \cdots$

(here as usual $\tau_\mathcal{F} = (\text{Id}_X \times \text{Frob}_S)^* \mathcal{F}$) with $\mathcal{F}_i$ vector bundles of rank $r$, such that $j$ and $t$ are $\mathcal{O}_X \times S$-linear maps satisfying

1. $\mathcal{F}_{i+r} = \mathcal{F}_i(\infty)$ and $j_{i+r} \circ \cdots \circ j_{i+1}$ is the natural map $\mathcal{F}_i \hookrightarrow \mathcal{F}_i(\infty)$.
2. $\mathcal{F}_i/j_i(\mathcal{F}_{i-1})$ is an invertible sheaf along $\Gamma_{\infty}$.
3. For all $i$, $\mathcal{F}_i/t_i(\tau_{i-1}) = \mathcal{F}_i^{-1}$ is an invertible sheaf along $\Gamma_z$ for some $z: S \rightarrow X \setminus \{\infty\}$ (independent of $i$).
4. For all geometric points $\mathfrak{p}$ of $\mathcal{S}$, the Euler characteristic $\chi(\mathcal{F}_0|_{\mathfrak{p}})$ vanishes.

If $I$ is a non-zero ideal of $A$, there is also a natural notion of $I$-level structure on an elliptic sheaf over $S$, at least if $S$ lives over $\text{Spec} A \setminus V(I)$, and Drinfeld proves the following remarkable result.

Theorem 4.2. Let $z: S \rightarrow \text{Spec} A \setminus V(I)$. Then there exists a bijection, functorial in $S$, between the two sets:

$$\left\{ \begin{array}{l} \text{rank } r \text{ elliptic } A\text{-modules over } S \\ \text{with } I\text{-level structure} \\ \text{such that } \phi(a)' = z(a) \end{array} \right\} / \cong \left\{ \begin{array}{l} \text{rank } r \text{ elliptic sheaves over } S \\ \text{with } I\text{-level structure} \\ \text{and zero } z \end{array} \right\} / \cong$$

1This is cheating a little: one has to apply carefully the Hochschild-Serre spectral sequence and one needs to introduce a compactification of $M_f^2$ to define the cuspidal cohomology of $M_f^2$ showing up on the left (corresponding to the space of cuspidal functions on the right).
The dictionary is explained in detail in [6] (see in particular the enlightening example $r = 1$ and its relation with geometric class field theory discussed there).

One shows that if $(\mathcal{F}, t, j)$ is an elliptic sheaf, then for all $i$,

$$t_i(\tau^*\mathcal{F}_{i-1}) = \mathcal{F}_i \cap t_{i+1}(\tau^*\mathcal{F}_i),$$

viewed as subsheaves of $\mathcal{F}_{i+1}$.

Hence, one can actually reconstruct the entire elliptic sheaf from the triangle

$$\begin{array}{c}
\mathcal{F}_0 \\
\mathcal{F}_1 \\
\tau^*\mathcal{F}_0
\end{array}$$

which is just a shtuka with two legs (one being fixed at $\infty$)! One can not go in the other direction – shtukas with two legs are more general than elliptic sheaves. There is no direct analogy anymore between shtukas with one pole at $\infty$ and one zero $z$ and elliptic curves (or abelian varieties) but the family of stalks at closed points of $X$ of such a vector bundle, with their Frobenius, behaves somehow like the family of $\varphi$-modules attached to the reduction mod $\ell$ of the $p$-divisible group of an abelian variety over a number field, when the prime $\ell$ varies (the choice of $\ell$ corresponding roughly to the choice of a closed point and the choice of $p$ corresponding to the choice of $z$). Shtukas with two legs are the right objects to consider to prove the full Langlands correspondence for $GL_r$ (for all $r$) over a function field, as demonstrated by [4], [5].

References