The Fargues-Fontaine curve and local Langlands: the case of GL_1 (after Fargues)

Automorphic Project & Research Seminar
(online)

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Reminder on geometric CFT:

X smooth projective geom. connected curve $/\mathbb{F}_q$
$E = \mathbb{F}_q(X)$.

Let $\mathbb{A} = \prod'_{x \in |X|} \hat{E}_x$, $\mathcal{O} = \prod_{x \in |X|} \hat{\mathcal{O}}_x$.

Unramified CFT tells us that the map

$$\mathbb{A}^\times \ni (a_x)_{x \in |X|} \mapsto \prod_{x \in |X|} \mathrm{Frob}_x^{\mathrm{ord}_x(a_x)} \in (\mathrm{Gal}_E^{unr})^{ab}$$

induces an isomorphism

$$\left(E^\times \backslash \mathbb{A}^\times / \mathcal{O}^\times \right)^\wedge \simeq (\mathrm{Gal}_E^{unr})^{ab}.$$

Would like to reformulate this statement geometrically. Note:

* $\mathrm{Gal}_E^{unr} \simeq \pi_1(X)$ (étale π_1)
* (Weil) $E^\times \backslash \mathbb{A}^\times / \mathcal{O}^\times \simeq \mathrm{Pic}_X(\mathbb{F}_q)$

Picard scheme of X

Hence, can reformulate the above as: Fix ℓ.
$\exists$ natural bijection

$$\left\{ \begin{array}{c} \text{continuous characters} \\ \pi_1(X) \to \overline{\mathbb{Z}}_\ell^\times \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{continuous characters} \\ \mathrm{Pic}_X(\mathbb{F}_q) \to \overline{\mathbb{Z}}_\ell^\times \end{array} \right\}.$$

s.t. if $\rho \mapsto \chi_\ell$, $\chi_\ell(\mathcal{O}(x)) = \rho(\mathrm{Frob}_x)$.

To go further observe:

* LHS $\longleftrightarrow$ rank 1 $\bar{\mathbb{Z}}_\ell$-local systems on X.

* RHS $\longleftrightarrow$ character loc. sys on Pic_X, i.e. $\bar{\mathbb{Z}}_\ell$-local systems on Pic_X s.t. $m^*\mathcal{F} \simeq p_1^*\mathcal{F} \otimes p_2^*\mathcal{F}$

$m : \mathrm{Pic}_X \times_{\mathbb{F}_q} \mathrm{Pic}_X \longrightarrow \mathrm{Pic}_X$ multiplication

p_1, p_2 projections. (Necessarily rank 1).

($\leftarrow$: take the trace of Frobenius.
$\rightarrow$: use the Lang isogeny, making Pic_X an étale torsor over itself with Galois gp $\mathrm{Pic}_X(\mathbb{F}_q)$.)

Rk: Character loc system:
homomorphism $G \longrightarrow B\Lambda^\times$ of commutative gp stacks
(rank 1 local system : morphism $G \to B\Lambda^\times$ of stacks).

Second (and final) reformulation:

Let $AJ : X \longrightarrow \mathrm{Pic}_X$ be the Abel-Jacobi map.
$x \longmapsto \mathcal{O}(x)$

Then: AJ^* induces an equivalence of categories

$$\left\{\begin{array}{c}\text{character local}\\ \text{systems on } \mathrm{Pic}_X\end{array}\right\} \simeq \left\{\begin{array}{c}\text{rank 1 local}\\ \text{systems on } X\end{array}\right\}.$$

Before giving the proof, some remarks:

(1) This reformulation now makes sense over any base field.

(2) In the above, used the Picard scheme Pic_X; could have used instead the Picard stack $\mathcal{P}ic_X$ (a $\mathbb{G}_m$-gerbe over Pic_X): ℓ-adic sheaves are the same. However in the following pf, will be important to work with Pic_X. The opposite will happen later in the local setting.

(3) Geometric approach can be extended to deal with ramification. (Guignard, Takeuchi)

<u>Pf (sketch)</u> (Deligne). Can work over $\overline{\mathbb{F}_q}$.

For $d \geq 1$, consider the degree d Abel-Jacobi map

$$AJ^d : X^{(d)} \longrightarrow \mathrm{Pic}_X^d, \quad D \mapsto \mathcal{O}(D).$$

$(AJ = AJ^1)$.

Let $\mathcal{L}$ be a rank 1 local system on X.

Because $\mathcal{L}$ has rank 1, $\mathcal{L}^{\boxtimes d} \in D_c^b(X^d, \overline{\mathbb{Q}_\ell})$ descends to a rank 1 local system $\mathcal{L}^{(d)}$ on $X^{(d)}$.

$\left(\text{Explicitly, } \mathcal{L}^{(d)} = (\pi_{d,*} \mathcal{L}^{\boxtimes d})^{S_d} \quad \pi_d : X^d \to X^{(d)}. \right)$

Fiber of AJ^d is the linear system
$$|D| = \mathbb{P}(H^0(X, \mathcal{O}(D))).$$
By Riemann-Roch, as long as $d > 2g-2$, this is fibration in projective spaces over Pic^d_X.

Fact: $\pi_1(\mathbb{P}^n_{\overline{\mathbb{F}_q}}) = 0$, $\forall n \geq 1$.

Hence, $\mathcal{L}^{(d)}$ descends to a local system $\mathrm{Aut}^d_{\mathcal{L}}$ on Pic^d_X for each $d > 2g-2$.

Since $\mathrm{Aut}^d_{\mathcal{L}}$ is obtained by descending $\mathcal{L}^{\boxtimes d}$, we deduce that for $d, d' > 2g-2$, if
$$m : \mathrm{Pic}^d_X \times \mathrm{Pic}^{d'}_X \longrightarrow \mathrm{Pic}^{d+d'}_X,$$
$$m^* \mathrm{Aut}^{d+d'}_{\mathcal{L}} \simeq \mathrm{Aut}^d_{\mathcal{L}} \boxtimes \mathrm{Aut}^{d'}_{\mathcal{L}}.$$
This property allows to define $\mathrm{Aut}^d_{\mathcal{L}}$ for all d and show that the local system $\mathrm{Aut}_{\mathcal{L}}$ so obtained is a character sheaf. ∎

local CFT: setting the stage

Assume now that E is a non-archimedean local field.
(i.e. $[E:\mathbb{Q}_p] < \infty$, $E \simeq \mathbb{F}_q((\pi))$)

Fargues: Can one replace the curve X in the above by "the" F-F curve for E to say something about CFT geometrically? $\left(\pi_1(X_{C,E}) = \mathrm{Gal}_E. \right)$

- C complete alg. closed non archimedean field over $\mathbb{F}_q$ $\rightsquigarrow$ $X_{C,E}$ depends on this auxiliary choice of C.
- $X_{C,E}$ is an adic space over $\mathrm{Spa}(E)$.

Consider "$\mathrm{Pic}_{C,E}$": $\left\{ \begin{array}{c} \text{adic spaces} \\ \text{over } E \end{array} \right\} \to \text{gpds}$

$$T \mapsto \left\{ \begin{array}{c} \text{line bundles on} \\ \text{on } X_{C,E} \times_{\mathrm{Spa} E} T \end{array} \right\}$$

makes sense at least for T rigid-analytic space.

Not many interesting ℓ-adic sheaves on "$\mathrm{Pic}_{C,E}$"!

(But still an interesting moduli problem, cf. Emerton-Gee, Hellmann)

Instead, better to keep the "E-direction fixed" and consider the "C-direction" as variable:

For any $S \in \mathrm{Perf}_{\mathbb{F}_q} = \{\text{perf'd spaces} / \mathbb{F}_q\}$ can define $X_{S,E}$ analytic adic space $/ E$, and consider the following moduli problems:

* Div^1_E: $S \mapsto \left\{ \begin{array}{l} \text{"relative Cartier divisors of} \\ \text{degree 1" on } X_{S,E} \end{array} \right\}$

* Pic_E $(= \mathrm{Bun}_1)$: $S \mapsto \left\{ \begin{array}{c} \text{line bundles on} \\ X_{S,E} \end{array} \right\}$.

<u>What is the geometry of these objects?</u>

1) Pic is a stack for the v-topology on $\mathrm{Perf}_{\mathbb{F}_q}$, with a simple geometric structure:

$$\mathrm{Pic} = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d \simeq \coprod_{d \in \mathbb{Z}} [* / \underline{E}^\times]$$

$(* = \mathrm{Spa}\,\mathbb{F}_q)$

any line bundle of degree d is v-locally isomorphic to $\mathcal{O}(d)$ and $\mathrm{Aut}(\mathcal{O}(d)) = \underline{E}^\times$.

NB: To see repr of $E^\times$, crucial to work with Picard stack, not Picard "scheme"!

In particular, $D_{\acute{e}t}(\mathrm{Pic}^d_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell) \simeq D\left(\begin{array}{c}\text{smooth } \overline{\mathbb{Q}}_\ell\text{-repr} \\ \text{of } E^\times\end{array}\right)$.

2) Better def^n of Div^1:

Let $S \in \mathrm{Perf}_{\mathbb{F}_q}$, $S^\#$ untilt of S / E.

$S = \mathrm{Spa}(R, R^+)$, $S^\# = \mathrm{Spa}(R^\#, R^{\#+})$

Fontaine's θ map: $W_{\mathcal{O}_E}(R^+) \longrightarrow R^{\#+}$

$\sum [x_n]\pi^n \mapsto \sum x_n^\# \pi^n$

surjective with kernel principal generated by a non zero-divisor ξ primitive of deg 1

i.e. $\xi \in W_{\mathcal{O}_E}(R)$, $x_0 \in R^\times \cap R^{\circ\circ}$, $x_1 \in (R^+)^\times$.
$= \sum [x_n]\pi^n$

This defines a map

$$\{\text{untilts over } E\}/\phi \longrightarrow \left\{\begin{array}{c}\text{closed Cartier} \\ \text{divisors in } Y_{S,E}\end{array}\right\}/X_{S,E}$$

Say that a closed Cartier divisor is relative of deg 1 if it is in the image of this map. Then,

$$\mathrm{Div}^1 \simeq \mathrm{Spa}\, E^\diamond/\phi^{\mathbb{Z}}.$$

Div^1 is a diamond.

<u>Rk</u>: $\mathrm{Div}^1 \longrightarrow *$ is proper & ℓ-cohom. smooth but Div^1 is not spatial (not quasi-separated).

Next, want to give another description of Div^1, better suited to the def^n & study of the Abel-Jacobi map.

Let G be the Lubin-Tate formal gp law with action of $\mathcal{O}_E$ over $\mathcal{O}_E$, G^{ad} its adic generic fiber. Since after choosing a coordinate, $G \simeq \mathrm{Spf}\,\mathcal{O}_E[[X]]$, $G^{ad} \simeq \mathring{\mathbb{D}}_E$ open unit disk $/E$.

Have a logarithm map: $\log_G : G^{ad} \longrightarrow \mathbb{A}^1_E$

$$X \longmapsto \sum_{i \geq 0} \pi^{-i} X^{q^i}$$

& exact sequence:

$$0 \to G^{ad}[\pi^\infty] \to G^{ad} \xrightarrow{\log_G} \mathbb{A}^1_E \to 0$$

(as étale sheaves over the big site for $\mathrm{Spa}\, E$).

Form the universal cover $\widetilde{G} = \varprojlim_{\times \pi} G \simeq \mathrm{Spf}\, \mathcal{O}_E[[\widetilde{X}^{1/q^\infty}]]$

$(\widetilde{X} = (X^{q^n})_n)$

Then, for any π-adically complete $\mathcal{O}_E$-alg A,

$$\widetilde{G}(A) = \widetilde{G}(A/\pi) \simeq \mathrm{Hom}_{\mathcal{O}_E}(E/\mathcal{O}_E, G(A/\pi))[\tfrac{1}{\pi}]$$

any element of $G(A/\pi)$ is killed by a power of π.

Also, $\widetilde{G}(A) = \mathrm{Hom}_{\mathcal{O}_E}(\mathcal{O}_E[[\widetilde{X}^{1/q^\infty}]], A) = \varprojlim_{x \mapsto x^q} A^{\circ\circ} = A^{\flat,\circ\circ}$.

Claim 1: $S = \mathrm{Spa}(R, R^+)$ aff'd perf $/\mathbb{F}_q$.

$S^\# = \mathrm{Spa}(R^\#, R^{\#+})$ untilt $/E$.

The map $\tilde{G}(R^{\#+}) = R^{\circ\circ} \to H^0(Y_{S,E}, \mathcal{O}_{Y_{S,E}})$

$$x \mapsto \sum_{i \in \mathbb{Z}} \pi^i [x^{q^{-i}}]$$

induces $\tilde{G}(R^{\#+}) \simeq H^0(Y_{S,E}, \mathcal{O})^{\varphi = \pi} \simeq H^0(X_{S,E}, \mathcal{O}(1))$.

Moreover, evaluation at $S^\#$:

$$H^0(X_{S,E}, \mathcal{O}(1)) \longrightarrow H^0(S^\#, \mathcal{O}_{S^\#}) = R^\#.$$

gets identified with

$$\tilde{G}(R^{\#+}) \longrightarrow G(R^{\#+}) \xrightarrow[\log_G]{} R^\#.$$

Pf. If $\mathrm{char}(E) > 0$, have

$$H^0(Y_{S,E}, \mathcal{O}) = \Big\{ \sum_{i \in \mathbb{Z}} x_i \pi^i, \ |x_i| r^i \underset{i \to \pm\infty}{\longrightarrow} 0 \ \ \forall\, 0 < r < 1 \Big\}$$

$(\ )^{\varphi = \pi} \Rightarrow x_i = x_{i+1}^q \ \forall i$

All determined by x_0, which must be top. nilp and can be any such an element.

If $[E : \mathbb{Q}_p] < \infty$, can use Dieudonné theory:

$$\mathrm{Hom}_{\mathcal{O}_E}(E/\mathcal{O}_E, G(R/p))[\tfrac{1}{\pi}] \simeq (B^+_{\mathrm{crys}})^{\varphi = \pi}$$

$$= B^{\varphi = \pi}.$$

Let $E_\infty = \left(\bigcup \{\pi^n\text{-torsion pts of } G \text{ in } \bar{E}\}\right)^{\wedge_\pi}$ perfectoid field. Over E_∞,

$$\mathcal{O}_E \simeq (T_\pi G)(\mathcal{O}_{E_\infty}) \subseteq \tilde{G}(\mathcal{O}_{E_\infty}).$$

Hence, by the claim, if $S^\#$ lives over E_∞, get a nonzero section of $\mathcal{O}(1)$ on $X_{S,E}$, vanishing at $S^\# \hookrightarrow X_{S,E}$.

<u>Claim</u> 2: This defines a short exact sequence:

$$0 \to \mathcal{O}_{X_{S,E}} \to \mathcal{O}_{X_{S,E}}(1) \to \mathcal{O}_{S^\#} \to 0.$$

This gives a map:

$$\big(\underbrace{BC(\mathcal{O}(1))\setminus\{0\}}\big)/\underline{E}^\times \longrightarrow \mathrm{Div}^1$$

($S \mapsto$ sections in $H^0(X_{S,E}, \mathcal{O}(1))$ fiberwise on S non-zero)

by sending such a section f to $V(f)$. It is well-defined: by Claim 1, $BC(\mathcal{O}(1)) \simeq \mathrm{Spd}\, \mathbb{F}_q[[x^{1/p^\infty}]]$, hence $BC(\mathcal{O}(1))\setminus\{0\} \simeq \mathrm{Spd}\, \mathbb{F}_q((x^{1/p^\infty}))$.

$$\simeq \mathrm{Spa}\, E_\infty^\flat = \mathrm{Spd}\, E_\infty.$$

Hence, gives by Claim 2 a map to Div^1, which is nothing but $\mathrm{Spd}\, E_\infty \to \mathrm{Spd}\, E \xrightarrow{\phi^{\mathbb{Z}}} \mathrm{Div}^1$.

$\underline{\mathcal{O}_E^\times}$ - torsor $\qquad \phi = \pi$ on $BC(\mathcal{O}(1)) \setminus \{0\}$.

Hence,

$$\mathrm{Div}^1 \simeq \left(BC(\mathcal{O}(1)) \setminus \{0\}\right) / \underline{E}^\times .$$

" A degree 1 Cartier divisor is the same thing as a deg 1 line bundle $\mathcal{L}$, fiberwise non-zero section u modulo $(\mathcal{L}, u) \sim (\mathcal{L}', u')$ if $\exists\, \underset{u \mapsto u'}{\mathcal{L} \simeq \mathcal{L}'}$. "

<u>Rk</u>: Note that $BC(\mathcal{O}(1)) \setminus \{0\} \simeq \mathrm{Spd}\, E_\infty^\flat \simeq \mathrm{Spd}\, \mathbb{F}_q((X^{1/q^\infty}))$ is a qcqs perf'd space but if C is a perfectoid field $/\mathbb{F}_q$, $BC(\mathcal{O}(1)) \setminus \{0\} \times_{\mathbb{F}_q} \mathrm{Spa}\, C \simeq \mathring{\mathbb{D}}^*_C$ isn't quasicompact anymore (i.e. $BC(\mathcal{O}(1)) \setminus \{0\} \to *$ is not qc)

<u>Rk</u>: The above identification can be made explicit on points, cf. Dospinescu's talk.

$$\underbrace{\left(\widetilde{G}(\mathcal{O}_C) \setminus \{0\}\right)}_{\mathfrak{m}_{C^\flat}} / \underline{E}^\times \underset{①}{\simeq} \mathrm{Div}^1(C) \underset{②}{\simeq} \left(BC(\mathcal{O}(1)) \setminus \{0\}\right) / \underline{E}^\times$$

① $\varepsilon \in \mathfrak{m}_{C^\flat} \longmapsto u_\varepsilon = \dfrac{[\varepsilon]_G}{[\varepsilon^{1/q}]_G}$, $[\cdot]_G = \lim\limits_{n \to \infty} [\pi^n]_G([x^{q^{-n}}])$

$[\varepsilon]_G = [\pi]_G [\varepsilon^{1/q}]_G$, $[\pi]_G = \begin{cases} \pi X & \text{mod deg} > 1 \\ X^q & \text{mod } \pi \end{cases}$ $\Rightarrow$ primitive of deg 1 in $A_{\mathrm{inf}}(\mathcal{O}_C)$.

② ξ primitive deg 1 $\longmapsto \Pi(\xi)$ "Weierstrass product"

assume $\xi \in \pi + W_{\mathcal{O}_E}(\mathfrak{m}_{C^\flat})$ $\qquad$ ॥

$$\underbrace{\Pi^+(\zeta)}_{=\prod_{n\geq 0}\frac{\varphi^n(\zeta)}{\pi^n}} \Pi^-(\zeta) \quad - \quad \varphi(\Pi^-(\zeta)) = \zeta\Pi^-(\zeta)$$

② ∘ ①: Can set $\Pi^-(u_\varepsilon) = \pi\,[\varepsilon^{1/q}]_q$.

& one computes: $\Pi^+(u_\varepsilon) = \frac{1}{\pi[\varepsilon^{1/q}]_q} \underbrace{\lim_{n\infty} \pi^{-n}[\pi^n]_q.[\varepsilon]_q}_{\log_q([\varepsilon]_q)}$

$\Rightarrow \Pi(u_\varepsilon) = \log_q([\varepsilon]_q)$, as expected!

More generally, define a <u>relative Cartier divisor of deg d</u> to be a line bundle $\mathcal{J}$ on X_S, together with an injective map $\mathcal{J} \hookrightarrow \mathcal{O}_{X_S}$, staying injective after pullback to any geom pt of S ("relative") s.t. $\mathcal{J}$ has degree d.

(By the above, did not change our def^n when $d=1$). Kedlaya-Liu $\Rightarrow$ moduli Div^d of deg d relative Cartier divisors is

$$\mathrm{Div}^d = (BC(\mathcal{O}(d))\setminus 0) / \underline{E}^\times.$$

Prop: The sum map $(\mathrm{Div}^1)^d \longrightarrow \mathrm{Div}^d$ induces $\mathrm{Div}^d \simeq (\mathrm{Div}^1)^d / S_d$.

(Both sides are proper v-sheaves, hence can check bijectivity on geom points, where it is Fargues-Fontaine factorization theorem "fundamental th. of (new) algebra".)

Rk: In particular, Div^d is a diamond (as $(\mathrm{Div}^1)^d \to \mathrm{Div}^d$ is finite hence quasi-pro-étale), and thus also $BC(\mathcal{O}(d)) \setminus \{0\}$ is. (In Dospinescu's talk, was only proved after b.c. to a perf'd space.)

Local CFT: geometric proof

Let $\rho : W_E \to \overline{\mathbb{Q}}_\ell^\times$ continuous character.

Since $\mathrm{Div}^1 = \mathrm{Spd}\, E/\varphi^{\mathbb{Z}}$, $\mathrm{Div}^1_k = \mathrm{Spd}\, \breve{E}/\varphi^{\mathbb{Z}}$
$(k = \overline{\mathbb{F}}_q)$ $\qquad = \mathrm{Spd}\, \hat{\overline{E}} / W_E$

hence ρ can be seen as a rank 1 local system $\mathcal{L}$ on Div^1_k.

Goal: descend $\mathcal{L}$ along

$$AJ = AJ_1 : \mathrm{Div}^1_k = (BC_k(\mathcal{O}(1)) \setminus 0)/\underline{E}^\times \to \mathrm{Pic}^1_k = [*/\underline{E}^\times]$$

If we can do that, we deduce that ρ comes from a character χ_ρ of $E^\times$, by composing χ_ρ^{-1} with the inverse of $\mathrm{Art}: E^\times \xrightarrow{\sim} W_E^{ab}$. Indeed, one has: $\mathrm{Art}^{-1}(g) = \chi_{LT}^{-1}(g)$ if $g \in I_E^{ab}$, $\mathrm{Art}^{-1}(\sigma) = \pi$ with $\sigma \in W_E^{ab}$ Frobenius, and as seen above, the $\underline{E}^\times$-torsor $BC_k(\mathcal{O}(1))\setminus\{0\} \to \mathrm{Div}^1_k$ identifies with

$$\mathrm{Spd}\, \breve{E}_\infty \longrightarrow \mathrm{Spd}\, \breve{E} \longrightarrow \mathrm{Spd}\, \breve{E}/\phi^{\mathbb{Z}}$$

Lubin-Tate $\underline{\mathcal{O}_E^\times}$-torsor $\qquad$ ϕ acts via π on $BC_k(\mathcal{O}(1))$

For $d \geq 1$, can again construct local system $\mathcal{L}^{(d)}$ on Div^d using $\mathrm{Div}^d \simeq (\mathrm{Div}^1)^d / S_d$.
Again, suffices to show that $\mathcal{L}^{(d)}$ is pulled back from a local system on Pic^d along

$$AJ^d : (BC_k(\mathcal{O}(d))\setminus\{0\})/\underline{E}^\times \longrightarrow \mathrm{Pic}^d_{k},$$

for some $d \gg 0$.
Reduces to prove the following statement:

<u>Theorem</u> (Fargues): For all $d \geq 3$,
$BC_k(\mathcal{O}(d))\setminus\{0\}$ simply connected.

<u>Rk</u>: Should be true for all $d \geq 2$.

Pf in char p: Assume $E \simeq \mathbb{F}_q((\pi))$.

Then (similarly as what happens when $d=1$)

$$BC(\mathcal{O}(d)) \simeq \mathrm{Spa}\, \mathbb{F}_q [\![x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]\!] \leftarrow \text{(not analytic)}$$

$S = \mathrm{Spa}(R, R^+)$

$$(R^{\circ\circ})^d \simeq H^0(Y_S, \mathcal{O})^{\varphi = \pi^d}$$

$$(x_0, \dots, x_{d-1}) \longmapsto \sum_{i=0}^{d-1} \sum_{k \in \mathbb{Z}} [x_i^{q^{-k}}] \pi^{i+kd}.$$

Thus, $BC(\mathcal{O}(d)) \setminus 0 \simeq \mathrm{Spa}(\mathbb{F}_q [\![x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]\!]) \setminus V(x_1, \dots, x_d)$

is a perfectoid space.

One gets:

$$\mathrm{FEt}(BC_k(\mathcal{O}(d)) \setminus 0) \simeq 2\text{-}\varinjlim \mathrm{FEt}(\mathrm{Spa}(k[\![x_1^{1/p^n}, \dots, x_d^{1/p^n}]\!]) \setminus V(x_1, \dots, x_d))$$

Hence, suffices to show that any finite étale cover of

$$\mathrm{Spa}(k[\![x_1, \dots, x_d]\!]) \setminus V(x_1, \dots, x_d)$$

splits.

Lemma: A noetherian ring, I ideal.

("GAGA") $\quad \mathrm{FEt}(\mathrm{Spa}(A) \setminus V(I)) \simeq \mathrm{FEt}(\mathrm{Spec}(A) \setminus V(I))$

Hence, $\widehat{\mathrm{FEt}}(\mathrm{Spa}\, k[[x_1,\dots,x_d]] \setminus V(x_1,\dots,x_d))$

$$\simeq \widehat{\mathrm{FEt}}(\mathrm{Spec}\, k[[x_1,\dots,x_d]] \setminus V(x_1,\dots,x_d)).$$

Zariski-Nagata $\Rightarrow$ $\simeq \widehat{\mathrm{FEt}}(\mathrm{Spec}\, k[[x_1,\dots,x_d]])$
$d \geq 2$

$$\underset{\text{Hensel}}{\simeq} \widehat{\mathrm{FEt}}(k)$$

$\blacksquare$

Rk: 1) The proof when $\mathrm{char}(E)=0$ is more complicated as then $BC(\mathcal{O}(d))\setminus\{0\}$, $d>1$, is merely a diamond, not a perf'd space. Still ultimately reduce to a purity result (in rigid geometry).

2) Above proof shows that for $E=\mathbb{F}_q((\pi))$, $d\geq 2$, $BC(\mathcal{O}(d))\setminus\{0\}$ is simply connected; but for any C complete alg closed perfectoid field $/\mathbb{F}_q$,

$$BC(\mathcal{O}(d))\setminus\{0\} \times_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\, C = \mathring{\tilde{\mathbb{D}}}^d_C$$

punctured perf'd open unit disk of dim d

which isn't simply connected!
(no Künneth formula for π_1 in char p.)

Beyond GL_1

What we achieved for GL_1: starting from $\rho : W_E \to \overline{\mathbb{Q}}_\ell^\times$, seen as a rank 1 local system $\mathcal{L}$ on Div^1_k, constructed $\mathrm{Aut}_{\mathcal{L}}$ local system on Pic_k with "character sheaf" property:

$$m^* \mathrm{Aut}_{\mathcal{L}} = p_1^* \mathcal{L} \otimes p_2^* \mathcal{L}$$

(by descending $\mathcal{L}^{(d)}$ to Pic^d_k for $d \gg 0$ and then extending to all of Pic_k as in the function field setting).

What about other groups?

$G = GL_n / E$ for simplicity (extends to all reductive groups $/E$).

Recall from Hansen's talk the v-stack

$$\mathrm{Bun}_n$$

on Perf_k of rank n v.b. on the FF-curve.

For each $b \in B(GL_n)$ ($\leftrightarrow$ rank n isocrystal) isoclinic, open substack $j_b : \mathrm{Bun}_n^b \simeq [*/\underline{G_b(E)}] \hookrightarrow \mathrm{Bun}_n$

(G_b : automorphism group of b).

Fargues conjectured in 2014 that to any indecomposable rep^n $\rho : W_E \longrightarrow GL_n(\overline{\mathbb{Q}_\ell})$ should be associated an object $\mathrm{Aut}_\rho \in D_{\acute{e}t}(\mathrm{Bun}_n, \overline{\mathbb{Q}_\ell})$

s.t. ① $\forall b \in B(GL_n)$ isoclinic,

$$j_b^* \mathrm{Aut}_\rho \longleftrightarrow LL_b(\rho) \circlearrowleft G_b(E)$$

$$D_{\acute{e}t}(\mathrm{Bun}_n^b, \overline{\mathbb{Q}_\ell}) \simeq D\left(\substack{\text{smooth} \\ \overline{\mathbb{Q}_\ell}\text{-repr of } G_b(E)}\right)$$

and s.t. ② Aut_ρ is a "Hecke eigensheaf with Hecke eigenvalue $\mathcal{L}$" ($\mathcal{L}$ rank n local system on Div^1_k corresponding to ρ).

The Hecke property implies Kottwitz conjecture saying that local Langlands and local Jacquet-Langlands are realized in the ℓ-adic cohomology of generalized Lubin-Tate spaces. Indeed, the latter can be reinterpreted as moduli spaces of modifications of vb on the FF-curve (Scholze-Weinstein).

When $n=1$, ① is the compatibility of the geometric constructions discussed above with Artin reciprocity (through Lubin-Tate theory).

② is saying that for any $d \geq 1$, if

$$\begin{aligned} m_d : \mathrm{Pic} \times (\mathrm{Div}^1)^d &\longrightarrow \mathrm{Pic} \\ (\mathcal{E}, x_1, \dots, x_d) &\longmapsto \mathcal{E}(\Sigma x_i), \end{aligned}$$

then $m_d^* \mathcal{F} \simeq \mathcal{F} \boxtimes \mathcal{L}^{\boxtimes d}$, which follows from the "character sheaf property".

Relation to the Kottwitz conjecture is in this case just the relation to Lubin-Tate theory.

[In terms of local shtukas: let $S = \mathrm{Spa}(R, R^+)$, $\xi \in W_{\mathcal{O}_E}(R^+)$ primitive of deg 1 ($\leftrightarrow (R^\#, R^{\#+})$). Set $\mathcal{E} = \mathcal{O}_{Y_S}$, and consider the shtuka $(\mathcal{E}, \varphi_\mathcal{E})$, with

$$\varphi_\mathcal{E} : \varphi^* \mathcal{E} \longrightarrow \mathcal{E} \quad \text{given by mult. by } \xi.$$

Then

$$\begin{aligned} \mathcal{E}_\infty &:= \varprojlim_{n \geq 0} \varphi^{n*} \mathcal{E} \\ \mathcal{E}^\infty &:= \varinjlim_{n \leq 0} \varphi^{n*} \mathcal{E} \end{aligned} \quad \varphi\text{-equivariant line bundles on } Y_S$$

with an injective morphism $u : \mathcal{E}_\infty \longrightarrow \mathcal{E}^\infty$
φ-equivariant
deduced from $\varphi_\mathcal{E}$.
Assuming $\xi \in \pi + W_{O_E}(R^{\circ\circ})$ (as we can up to renormalizing), $\Pi^+(\xi) = \prod_{n\geq 0} \frac{\varphi^n(\xi)}{\pi}$ well-def
and defines a non-zero section of $\mathcal{E}_\infty(1)$, i.e.
a trivialization $\mathcal{E}_\infty \simeq \mathcal{O}(-1)$. On the other hand, there is up to multiplication by an element of $E^\times$, a unique $\Pi^-(\xi)$ s.t. $\varphi(\Pi^-(\xi)) = \xi\, \Pi^-(\xi)$, defining a trivialization $\mathcal{E}^\infty \simeq \mathcal{O}$. Then $\Pi(\xi) = \Pi^+(\xi)\Pi^-(\xi)$
give a trivialization of the Tate module of the p-divisible group over $R^{\#+}$ attached to $(\mathcal{E}, \varphi_\mathcal{E})$.]