A Fourier transform for Banach-Colmez spaces

King's College London

May 2022

Let $k = \mathbb{F}_q$ finite field of char p.
Choose $\psi : k \to \overline{\mathbb{Q}}_\ell^\times$ ($\ell \neq p$) non-trivial character.
The Artin-Schreier torsor ("Lang isogeny"): $\mathbb{A}^1_k \xrightarrow{\mathbb{A}^1(k)} \mathbb{A}^1_k$
allows us to attach to ψ a local system $\mathcal{L}_\psi \in D_{ét}(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell)$
and to consider the functor (Deligne-Laumon):

$$\mathcal{F}_\psi : D_{ét}(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell) \longrightarrow D_{ét}(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell)$$

$$A \longmapsto \check{\pi}_!(\pi^* A \otimes m^* \mathcal{L}_\psi)$$

$$\begin{tikzcd} & \mathbb{A}^1_k \times \mathbb{A}^1_k \arrow[r, "m"] \arrow[dl, "\check{\pi}"'] \arrow[dr, "\pi"] & \mathbb{A}^1_k \\ \mathbb{A}^1_k & & \mathbb{A}^1_k \end{tikzcd}$$

More generally, we could have done this over any scheme/stack S over $\mathrm{Spec}(k)$ (instead of $\mathrm{Spec}(k)$) and any geometric vector bundle $\mathcal{V}$ over S (instead of $\mathbb{A}^1_k$), using the duality pairing $\mathcal{V}^\vee \times_S \mathcal{V} \to \mathbb{A}^1_S$, to get a functor

$$\mathcal{F}_{\psi,\mathcal{V}} : D_{ét}(\mathcal{V}, \overline{\mathbb{Q}}_\ell) \longrightarrow D_{ét}(\mathcal{V}^\vee, \overline{\mathbb{Q}}_\ell)$$

This functor enjoys nice properties: it commutes with any base change in S, its composition with itself is "mult. by -1", it commutes with Verdier duality (all up to shift/twist).

<u>Rk</u>: Generalizes usual FT via sheaf/functions dictionary.

Now, what if we replace k by E, finite extension of $\mathbb{Q}_p$?

Obvious problem: E has characteristic 0, so

$$\pi_1^{\acute{e}t}(\mathbb{A}^1_E) = 0,$$

so no kernel attached to the choice of $\psi : E \to \overline{\mathbb{Q}}_\ell^\times$!

Remedy: Things get better if we replace the scheme $\mathbb{A}^1_E$ by the adic affine line $\mathbb{A}^{1,ad}_E$ (denoted simply $\mathbb{A}^1_E$ in the following).

Indeed, let G be the Lubin-Tate formal gp law of E. (1-dim'l formal gp with action of $\mathcal{O}_E$). As a formal scheme, $G = \mathrm{Spf}\ \mathcal{O}_E[[X]]$, and the logarithm of G defines a surjective étale morphism of adic spaces:

$$\log_G : \underbrace{G_E^{ad}}_{\text{adic generic fiber}} \simeq \underset{\substack{\uparrow \\ \text{rigid analytic} \\ \text{open disk over } E}}{\mathbb{D}_E} \longrightarrow \mathbb{G}_{a,E}^{ad} \simeq \mathbb{A}^1_E.$$

with kernel $G_E^{ad}[\pi^\infty]$ (π uniformizer of E).

Passing to the universal cover, this gives a short exact sequence of sheaves on the big $^{(\text{pro-})}$étale site of E:

$$0 \to V_\pi G \longrightarrow \widetilde{G_E^{ad}} = \mathbb{D}_E^\infty \longrightarrow \mathbb{A}^1_E \to 0.$$

After base change to the LT-extension

$$E_\infty := \text{completion of } \bigcup_{n\geq 0} E[\pi^n],$$

$V_\pi G \simeq \underline{E}$ and thus we get a pro-étale $\underline{E}$-torsor over $\mathbb{A}^1_{E_\infty}$. Hence, using it and choosing a non-trivial smooth character $\psi : E \to \overline{\mathbb{Q}}_\ell^\times$, we can define

$$\mathcal{F}_\psi : D_{et}(\mathbb{A}^1_{E_\infty}, \overline{\mathbb{Q}}_\ell) \to D_{et}(\mathbb{A}^1_{E_\infty}, \overline{\mathbb{Q}}_\ell).$$

enjoying the same formal properties as the ℓ-adic FT of Deligne-Laumon.

To go further, it will be useful to reinterpret the above short exact sequence of pro-étale sheaves (over $\mathrm{Spa}\, E_\infty$)

$$0 \to \underline{E} \to \widetilde{G}_E^{ad} \to \mathbb{A}^1_E \to 0 \qquad (*)$$

in terms of the Fargues-Fontaine curve.

Recall: F perf'd field of char $p \supseteq k$ (residue field of E)
(e.g. $F = \hat{\overline{E}}^\flat$)

$\rightsquigarrow X_{F,E} \to \mathrm{Spa}(E)$ adic space "parametrizing untilts of F over E":

$$= \left(\mathrm{Spa}\left(W_{\mathcal{O}_E}(\mathcal{O}_F), W_{\mathcal{O}_E}(\mathcal{O}_F)\right) \setminus V(\pi[\varpi])\right) / \phi^{\mathbb{Z}}.$$

If (D, φ_D) isocrystal over k_F, get an associated vector bundle $\mathcal{E}(D, \varphi_D)$ over $X_{F,E}$.

The line bundle $\mathcal{O}(1)$ attached to the isocrystal (E, π^{-1}) sits in an exact sequence, when $F = \hat{E}^b$,

$$0 \to \mathcal{O} \to \mathcal{O}(1) \to i_{\infty,*} \hat{E} \to 0$$

rigid pt of $X_{F,E}$ corresponding to until $\hat{E}$ of F.

Taking global sections of this exact seq, get a short exact seq of E-VS:

$$0 \to E \to (B^+_{\mathrm{crys},E})^{\varphi = \pi} \to \hat{E} \to 0$$

which corresponds to the evaluation on $\mathrm{Spa}\,\hat{E}$ of $(*)$.

(Key pt: $\tilde{G}(\mathcal{O}_{\hat{E}}) = \tilde{G}(\mathcal{O}_{\hat{E}}/\pi)$

(any element in $G(\mathcal{O}_F/\pi)$ is π-torsion) $= \mathrm{Hom}_{\mathcal{O}_E}(E/\mathcal{O}_E, G(\mathcal{O}_{\hat{E}}/\pi))[\frac{1}{\pi}]$

$= (B^+_{\mathrm{crys},E})^{\varphi = \pi}$ (Scholze-Weinstein))

but we can do better: the construction of the curve makes sense for any perf'd space S over $\mathrm{Spa}\,k$

(instead of just $\mathrm{Spa}(\mathbb{F})$). There is no morphism $X_{S,E} \to S$, but the association $S \mapsto X_{S,E}$ is functorial in S and sends pro-étale covers to pro-étale covers, hence define a morphism of sites

$$\tau : (X_{S,E})_{\text{proét}} \longrightarrow \underbrace{S_{\text{proét}}}_{\substack{\text{site of all perf'd spaces} \\ \text{over } S \text{ (big site)}}}$$

for any $S \in \mathrm{Perf}_{\mathbb{F}_q}$.

In particular, get a functor

$$R\tau_* : \underbrace{\mathrm{Perf}(X_{S,E})}_{\substack{\text{perfect complexes} \\ \text{over } X_{S,E}}} \longrightarrow D(S_{\text{proét}}, \underline{E})$$

s.t. when $S = \mathrm{Spa}(E_\infty^\flat)$, $R\tau_*$ applied to

$$0 \to \mathcal{O} \to \mathcal{O}(1) \to i_{\infty,*}E_\infty \to 0$$

recovers the exact sequence of pro-étale sheaves $(*)$.

(via $\mathrm{Spa}(E_\infty^\flat)_{\text{proét}} \simeq \mathrm{Spa}(E_\infty)_{\text{proét}}$)

From this perspective, the $\underline{E}$-torsor over $\mathbb{A}^1_{E_\infty}$ furnished by LT-theory comes from the fact that

$$\mathrm{Ext}^1_{X_{E_\infty^\flat,E}}(i_{\infty *}E_\infty, \mathcal{O}) \neq 0$$

and more precisely $\underline{\mathrm{Ext}}^1_{X_{E_\infty^\flat,E}}(i_{\infty,*}E_\infty, \mathcal{O}) \simeq i_{\infty,*}E_\infty$.

Using this perspective, can extend Ramero's FT to a more general class of objects: for any (flat) coherent sheaf $\mathcal{E}$ on $X_{S,E}$ (= 2-term complex with only coh. in deg 0) having only positive slopes,

$$R\Gamma_{\text{ét}}\,\mathcal{E} = R^0\Gamma_{\text{ét}}\,\mathcal{E} =: BC(\mathcal{E})$$

one can define a "dual" $BC(\mathcal{E})^\vee$ with a natural pairing $BC(\mathcal{E})^\vee \times BC(\mathcal{E}) \xrightarrow{\alpha} [S/\underline{E}]$, and an associated FT, recovering Ramero's for $S = \mathrm{Spa}(E_\infty)$, $\mathcal{E} = i_{\infty,*}E_\infty$.

Rk: 1) For technical reasons, need to work with torsion (prime to p) coefficients, but will ignore this and keep $\overline{\mathbb{Q}}_\ell$-coeffs.

2) Can also allow ss vector bundles of slope 0 in the formalism, and in this case the FT is very closely related to the usual FT for smooth functions on f.d. E-vs.

3) v-descent results allow the base S to be an arbitrary "small v-stack".

For the rest of this talk, would like to discuss one specific example.

From now on, all sheaves and stacks will be considered over $\mathrm{Spa}\,\bar{k}$.

Let $S = \mathrm{Pic}_E^1$ be the degree 1 component of the Picard stack $\mathrm{Pic}_E : T \in \mathrm{Perf} \mapsto \{\text{line bundles on } X_{T,E}\}$, $\mathcal{L}$ universal line bundle over S.

Kedlaya-Liu: $S \simeq [\mathrm{Spa}\,\bar{k} / \underline{E}^\times]$ (possible locally on S, any deg 1 line bundle is $\simeq \mathcal{O}(1)$ on $X_{S,E}$)

$$BC(\mathcal{L}) \simeq [BC(\mathcal{O}(1)) / \underline{E}^\times]$$

Moreover, $BC(\mathcal{L})^\vee = [BC(\mathcal{O}(-1)) / \underline{E}^\times]$.

Hence in this case, the Fourier transform is an equivalence

$$\mathcal{F}_\psi : D_{et}\left(\frac{BC(\mathcal{O}(1))}{\underline{E}^\times}, \Lambda\right) \simeq D_{et}\left(\frac{BC(\mathcal{O}(-1))}{\underline{E}^\times}, \Lambda\right)$$

What do both sides look like?

* Recall, as mentioned above, that

$$BC(\mathcal{O}(1)) = \widetilde{\mathcal{G}}_E^{ad} \quad (\mathcal{G} \text{ Lubin-Tate f.g. law})$$

If $R \in \mathrm{Perf}_{\bar{k}}$, with untilt $R^\#$ over E,

$$BC(\mathcal{O}(1))(\mathrm{Spa}(R,R^+)) = \widetilde{\mathcal{G}}(R^{\#+}) \simeq \varprojlim_{x \mapsto x^q} R^{\#,\circ\circ} \simeq R^{\circ\circ}.$$

Hence, $BC(\mathcal{O}(1)) = \mathrm{Spa}\,\bar{k}[\![t^{1/p^\infty}]\!]$.

More canonically, $BC(\mathcal{O}(1)) \simeq \mathrm{Spd}(\mathcal{O}_{\breve{E}_\infty})$

and the actions of E on both sides identify.

In particular, $BC(\xi) \hookrightarrow_j \dfrac{(BC(\mathcal{O}(1)) \setminus \{0\})}{\underline{E}^\times} = \dfrac{\mathrm{Spd}\, \breve{E}}{\phi^{\mathbb{Z}}}$

$\Rightarrow$ local $\overline{\mathbb{Q}}_\ell$-sheaves on LHS $\leftrightarrow$ continuous $\overline{\mathbb{Q}}_\ell$-reps of W_E.

$*$ $BC(\mathcal{O}(-1))$ is the moduli of extensions

$$0 \to \mathcal{O} \to \mathcal{F} \to \mathcal{O}(1) \to 0.$$

$$\Rightarrow \quad BC(\mathcal{O}(-1)) \setminus \{0\} = \left(BC(\mathcal{O}(\tfrac{1}{2})) \setminus \{0\}\right) \Big/ \ker(\mathrm{Nrd} : D^\times \to E^\times)$$

Here, $D = \mathrm{End}(\mathcal{O}(\frac{1}{2}))$ quaternion alg over E.

Hence, $BC(\xi)^\vee \hookrightarrow_{j^\vee} \dfrac{BC(\mathcal{O}(-1)) \setminus \{0\}}{\underline{E}^\times} = \dfrac{BC(\mathcal{O}(\frac{1}{2})) \setminus \{0\}}{\underline{D}^\times}$.

Let $\sigma : W_E \to GL_2(\overline{\mathbb{F}}_\ell)$ continuous *irreducible*, $\mathbb{L}$ local system on $\dfrac{BC(\mathcal{O}(1)) \setminus \{0\}}{\underline{E}^\times}$ corresponding to it.

Conj: $j^{\vee *} \mathcal{F}_\varphi\, j_! \mathbb{L}[1] \in D_{et}\left(\dfrac{BC(\mathcal{O}(\frac{1}{2})) \setminus \{0\}}{\underline{D}^\times}, \overline{\mathbb{F}}_\ell\right)$

comes via pullback along

$$\underline{\mathrm{BC}(\mathcal{O}(\tfrac{1}{2}))\mathrm{Isof}}/\underline{D}^\times \longrightarrow [\mathrm{Spa}\,\bar{k}/\underline{D}^\times] \leftarrow \begin{array}{c} D\left([\mathrm{Spa}\bar{k}/\underline{D}^\times], \bar{\mathbb{Q}}_\ell\right) \\ \cong \\ D(\text{smooth } \bar{\mathbb{Q}}_\ell\text{-reps of } D^\times) \end{array}$$

from the smooth rep $\pi(\sigma)$ of $D^\times$ attached to σ by local Langlands + Jacquet-Langlands.

As a weak evidence towards this, let us compute dimensions.

Since $D^\times$ is compact mod center, $\pi(\sigma)$ has finite dimension, which was computed by Carayol:

$$\dim \pi(\sigma) = \begin{cases} 2q^{\mathrm{sw}(\sigma)/2} & \mathrm{sw}(\sigma) \text{ even} \\ (q+1)\,q^{\frac{\mathrm{sw}(\sigma)}{2}-1}, & \text{— odd.} \end{cases}$$

(assuming $\mathrm{sw}(\sigma)$ can't be lowered by twisting by a character)

Pick $x: \mathrm{Spa}(C) \to \mathrm{BC}(\mathcal{O}(-1))\mathrm{Isof}/\underline{E}^\times$ a geometric pt. ($C/\mathbb{F}_q$ complete alg closed). By proper base change,

$$\mathcal{F}_\varphi(j_!\mathbb{L})_x \simeq R\Gamma_c(\mathbb{D}^*_C, (\tilde{\mathbb{L}} \otimes \alpha^*\mathcal{L}_\varphi)_C),$$

with $\tilde{\mathbb{L}}$ = pullback of $\mathbb{L}$ to $\mathrm{BC}(\mathcal{O}(1))\mathrm{Isof}$ ($\leftrightarrow$ restriction to $\mathrm{Gal}^{u}_{E_\infty}$).

Can be shown to be a perfect complex (using that everything is defined "absolutely" over $\mathrm{Spa}\,k$), concentrated in deg 1.

What is its Euler characteristic?

Huber's nice observation: the theory of Swan conductors extend to reps of G, Galois gp of a Henselian valued field K with value gp $\Gamma \simeq \Gamma_{div} \times \mathbb{Z}$.

Ex: • K discretely valued, $\Gamma_{div} = \{1\}$.

• K = henselization of the residue field at a rank 2 (type 5) point of a smooth adic curve over C.

$\Gamma = \Gamma_C \times \mathbb{Z}$, with lexicographic order.

(Ex: $r \in (0,\infty)$, $x_{<r}$ and $x_{>r}$ two rank 2 points of "radius r" in $\mathbb{A}^1_C$.)

Th (Huber): Let A be a $\overline{\mathbb{Q}}_\ell$-local system on $\mathbb{D}^\circ_C$, s.t. $R\Gamma_c(\mathbb{D}^\circ_C, A)$ is a perfect complex. Then:

$$\chi\left(R\Gamma_c(\mathbb{D}^\circ_C, A)\right) = -\mathrm{sw}_{x<r}(A) - \mathrm{sw}_{x>r'}(A)$$

for any r close to 0 and r' close to 1.

We therefore have to analyze such conductors for

$$A = (\mathbb{L} \otimes \alpha^* \mathcal{L}_\psi)_C.$$

Around 0, $(\alpha^*\mathcal{L}_\psi)_C$ is tamely ramified (as it extends to $\mathbb{D}_C$) and the contribution of $\mathbb{L}$ dominates:

$\forall r$ small, $\mathrm{sw}_{x<r}(A) = \mathrm{rank}((\alpha^*\mathcal{L}_\psi)_C) \cdot \mathrm{sw}_{x<r}(\mathbb{L}_C) = \mathrm{sw}_{x<r}(\mathbb{L}_C)$.

Using Fontaine–Wintenberger's ramification estimates in their fields of norm theory and the computation of the inverse Herbrand function for the LT-ext., get:

$$\mathrm{sw}_{x<r}(\widetilde{\mathbb{L}}_C) = \begin{cases} 2\left(q^{\mathrm{sw}(\sigma)/2} - 1\right), & \mathrm{sw}(\sigma) \text{ even} \\ (q+1)q^{\frac{\mathrm{sw}(\sigma)-1}{2}} - 2, & \text{— odd.} \end{cases}$$

Close to 1, the contribution of $(\alpha^* \mathcal{L}_\psi)_C$ dominates:

$$\mathrm{sw}_{x>r'}(A) = \mathrm{rank}(\mathbb{L}_C) \cdot \mathrm{sw}_{x>r'}((\alpha^*\mathcal{L}_\psi)_C)$$

r' close to 1

$$= 2 \cdot 1 = 2.$$

Hence $\chi(R\Gamma_c(\mathbb{D}^*_C, A)) = -\dim \pi(\sigma)$,

as expected.