Banach-Colmez spaces

Stockholm

June 2022

Let C be a complete alg. closed non-archimedean field over $\mathbb{Q}_p$, $C^\flat$ its tilt.

We let $(\mathrm{Spa}\, C)_v$ be the v-site of $\mathrm{Spa}\, C$: the category of all perf'd spaces over $\mathrm{Spa}\, C$, endowed with the v-topology (topology generated by surjective morphisms $S' \to S$ of aff'd perf'd spaces).

<u>Def</u>: The <u>category of Banach-Colmez spaces</u> is the smallest abelian subcategory stable by extensions and containing the v-sheaves $\underline{\mathbb{Q}_p}$ and $\mathbb{G}_{a,C}$ of the category of sheaves of $\mathbb{Q}_p$-modules on $(\mathrm{Spa}\, C)_v$.

Here, $\underline{\mathbb{Q}_p} : S \mapsto \mathrm{Cont}(|S|, \mathbb{Q}_p)$

$\mathbb{G}_{a,C} := (\mathbb{A}^1_C)^\diamond : S \mapsto \mathcal{O}_S(S)$.

This definition is reminiscent of the definition of unipotent perfect grp schemes in char. p:

Let k alg. closed field of char. p. The category of

$\underbrace{\text{perfect algebraic}}_{(\simeq G^{pf},\ G\ \text{affine})}$ group schemes killed by p over k is an abelian subcategory stable by extensions of the category of sheaves of $\mathbb{F}_p$-VS on the perfect étale/v-site of $\mathrm{Spec}(k)$, and any object has a composition series with quotients isomorphic to $\mathbb{G}_a$ or $\mathbb{F}_p$.

In this char. p setting more is known:

- The functor $D: \mathcal{G} \mapsto R\underline{\mathrm{Hom}}_{D((\mathrm{Spec}\,k)_v, \mathbb{F}_p)}(\mathcal{G}, \mathbb{F}_p)$ defines an auto-duality of the subcategory $\mathcal{D}(\mathrm{Spec}(k))$ of $D((\mathrm{Spec}\,k)_v, \mathbb{F}_p)$ formed by bdd complexes with cohomology in the above abelian category.

  E.g. if $\mathcal{G} = G[0]$, G étale, $D(\mathcal{G}) = \underset{\substack{\uparrow \\ \text{Pontryagin dual}}}{G^*}[0]$

  if $\mathcal{G} = G[0]$, G connected
  $$D(\mathcal{G}) = \underline{\mathrm{Ext}}^1_{D(\mathrm{Spec}\,k, \mathbb{F}_p)}(G, \mathbb{F}_p)[-1].$$

- Can extend the definition of $\mathcal{D}(\mathrm{Spec}\,k)$ to the relative setting: Kato did it in the 80's for smooth varieties over a char p perfect field.

This is useful to study duality: if X smooth proper variety over $\mathrm{Spec}(k)$, the flat cohomology of μ_p on X $R\Gamma_{flat}(X, \mu_p)$ can naturally be seen as an object of $\mathcal{D}(\mathrm{Spec}\,k)$ and satisfies duality w.r.t. $\mathbb{D}$.

What about analogues of these results for Banach-Colmez spaces?

Rather surprisingly, describing non-trivial examples of BC spaces involves Fontaine's period rings. E.g.:

$$0 \to \mathbb{G}_{a,C} \to \mathbb{B}^+_{dR,C}/t^2 \to \mathbb{G}_{a,C} \to 0$$

$$0 \to \underline{\mathbb{Q}_p} \to (\mathbb{B}^+_{crys})^{\varphi = p} \to \mathbb{G}_{a,C} \to 0$$

non-split extensions.

In fact the category of Banach-Colmez spaces is closely related to the category of coherent sheaves on the FF-curve attached to $\mathbb{Q}_p$ and $C^\flat$.

Let E local field of res. char p, $\mathbb{F}_q$ = res. field, unif π.

$S \in \mathrm{Perf}_{\mathbb{F}_q}$ $\quad X_S = Y_S/\varphi_S$, $\quad Y_S = \mathrm{Spa}(W_{\mathcal{O}_E}(R^+)) \setminus V(\pi[\varpi])$

attached to E, fixed. ↗ $\quad$ if $S = \mathrm{Spa}(R, R^+)$ with p.u. ϖ.

Th 1: The functor $S \mapsto \mathcal{P}erf(X_S)$ is a v-sheaf of ∞-categories

cat. of perfect complexes on X_S

Rk: a) For vector bundles, the result is already known and due to Kedlaya-Liu. Critical to define Fargues-Scholze's Bun_G (G reductive$/E$).

b) Our motivation was to define a good notion of "flat coherent sheaves on X_S" satisfying v-descent in S. Here: $\mathcal{O}_{X_S}$-module which can, locally for the analytic topology on X_S, be written as the cokernel of a morphism b/w vector bundles which remain injective after base changing along all geom pts of S.

Proof (sketch):

Step ①:

Let $E_\infty = \widehat{E(\pi^{1/p^\infty})}$. Using that the map $E \to E_\infty$ splits as a map of E-Banach spaces and that the adic space $X_S \times_{\mathrm{Spa}\, E} \mathrm{Spa}\, E_\infty$ is perf'd, Th 1 is reduced to:

Sketch of pf: Let $f: T \to S$ be a v-cover.
Let $T^{\bullet/S} \to S$ Čech nerve of f
(simplicial perf'd space)

Need to show: $\Phi: \mathrm{Perf}(S) \to \varprojlim_{\Delta} \mathrm{Perf}(T^{\bullet/S})$

is an equivalence.

One important input in the proof is the following result of Andreychev.

Th (Andreychev) The functor $\mathrm{Spa}(R,R^+) \mapsto \mathrm{Perf}(R)$ (perfect complexes of R-modules) on the category of analytic affinoid adic spaces is a sheaf for the analytic topology.

The proof goes by identifying $\mathrm{Perf}(X)$ with dualizable objects in a certain symm monoidal category of solid qcoh sheaves on X.

In particular, can assume $S = \mathrm{Spa}(R,R^+)$, $T = \mathrm{Spa}(R',R'^+)$.
Fully faithfulness of Φ follows from exactness of

$$0 \to R \to R' \to R' \hat{\otimes}_R R' \to \cdots$$

and dévissage.

The hard part is essential surjectivity. We follow the following strategy:

First, when $(L, L^+) = (k, k^+)$ perf'd field, result can be easily deduced from the case of v.b. by truncations.

Def: $\mathrm{Spa}(A, A^+)$ affinoid space, $M \in \mathrm{Perf}(A)$. An <u>integral model</u> of M is a perfect complex $M^+ \in \mathrm{Perf}(A^+)$ s.t. $M^+ \otimes^{\mathbb{L}}_{A^+} A \simeq M$.

Back to S: let $N^\bullet \in \varprojlim_{\Delta} \mathrm{Perf}(T^{\bullet/S})$.

Pick $x \in S$. By analytic descent, enough to show $N^\bullet$ descends in a neighborhood of x. By the case of fields, restriction $N^\bullet_x$ of $N^\bullet$ to $f^{-1}(x)^{\bullet/Sx}$ descends to some $M_x \in \mathrm{Perf}(k(x))$. If we choose an integral model of M_x, get integral model $N^{\bullet,+}_x$ of $N^\bullet_x$ by pullback.

<u>Lemma</u>: (A, A^+) complete uniform Tate-Huber pair. $M \in \mathrm{Perf}(A)$. The functor F_M:

$$(B, B^+) \longmapsto \{\text{int. models of } B \otimes^{\mathbb{L}}_A M\}$$

complete uniform Tate-Huber pair over (A, A^+)

commutes with filtered colimits (which are completed!).

Pf: (B_i, B_i^+), $i \in I$, filtered system.
(B, B^+) uncompleted colimit, $(C, C^+) = (B, B^+)^\wedge$.

Scholze: the diagram
(derived version of Beauville-Laszlo)

$$\begin{tikzcd} \mathrm{Perf}(B^+) \arrow[r] \arrow[d] & \mathrm{Perf}(B) \arrow[d] \\ \mathrm{Perf}(C^+) \arrow[r] & \mathrm{Perf}(C) \end{tikzcd}$$

is cartesian $\Longrightarrow$ $\mathcal{F}_M(C, C^+) = \mathrm{Perf}(B^+) \times_{\mathrm{Perf}(B)} \{M \overset{L}{\underset{A}{\otimes}} B\}$

$$\simeq \varinjlim_i \mathcal{F}_M(B_i, B_i^+). \blacksquare$$

Since $(k(x), k(x)^+)$ filtered colimit in complete uniform Tate-Huber pairs of rational open neighborhood of x in S, can up to shrinking S, assume that we have an integral model $N^{\circ,+} \in \varprojlim_{\Delta} \mathrm{Perf}(R_n'^+)$ for N°.

On the other hand, can also extend the integral model M_x^+ to an integral model M^+ of M (again, up to shrinking S).

The descent data N^+/π^k and $\widetilde{M^+/\pi^k}$ (deduced from M^+/π^k by pullback to T'/S) become isomorphic on the fiber $f^{-1}(x) = \varprojlim_{U \ni x} T \times_S U$.

$\Rightarrow$ Up to shrinking S, $\widetilde{M}^+/^{\mathbb{L}}\pi \simeq N^+/^{\mathbb{L}}\pi$.

By induction on n, define $M_n^{+,a}$ unique descent of $N^{+,\bullet,a}/^{\mathbb{L}}\pi^n$ $\overset{\cap}{\mathrm{Perf}}(R^{+,a})$

(only almost, as integrally Φ is only fully faithful on almost categories)

These are compatible and can set $M^{+,a} = \varprojlim_n M_n^{+,a}$.

Then $M := M^{+,a}[\frac{1}{\pi}]$ defines a dualizable object in $D((R,R^+)_\bullet)$, hence (Andreychev) gives a perfect complex of R-modules descending $N^\bullet$. ∎

Recall that the relative FF-curve X_S, although it can be thought of as a family of "usual" FF-curves attached to geom. pts of S, does <u>not</u> live over S, as an adic space. However, have a morphism of sites $\tau : (X_S)^\diamond_v \longrightarrow S_v$.

<u>Th 2</u>: $R\tau_* : \mathrm{Perf}(X_S) \longrightarrow D(S_v, \underline{E})$ is fully faithful.

Th 1': $S \mapsto \mathrm{Perf}(S)$ is a v-stack (of ∞-cat) on the category of all perf'd spaces.

Step ② : Prove descent for a v-cover $\mathrm{Spa}(L,L^+) \to \mathrm{Spa}(K,K^+)$, K, L perfectoid fields (reduce by truncations to v-b.)

Step ③ : Andreychev has proved analytic descent of perfect complexes. Hence can prove descent analytic-locally. Then use that for $s \in S$,

$$\mathrm{Spa}(k(s),k(s)^+) = \varprojlim_{U \ni s} U$$

+ spreading out argument. ■

Recall that the relative FF-curve X_S, although it can be thought of as a family of "usual" FF-curves attached to geom.-pts of S, does not live over S, as an adic space. However, have a morphism of sites $\tau : (X_S)^\diamond_v \longrightarrow S_v$.

Ex: • $R\tau_* \mathcal{O} = \underline{E}[0]$.

• $S^\#$ untilt of S over E. Defines a "Cartier divisor" $i_{S^\#} : S^\# \to X_S$ and

$$R\tau_*(i_{S^\#,*}\mathcal{O}_{S^\#}) = \mathbb{G}_{a,S^\#}[0].$$

Th 2: $R\tau_* : \mathrm{Perf}(X_S) \to D(S_v, \underline{E})$ is fully faithful.

Rks: 1) Rather surprising: false for $\mathbb{P}^1$! ($\mathcal{O}(-1)$ has no cohomology)

2) A mod p analogue of this (and much more) has been proved by Mann:

dualizable objects in $D^a_\square(S, \mathcal{O}_S^+/\pi)^\varphi \hookrightarrow D(S_v, \mathbb{F}_p)$

weakly almost perfect complexes of

$\overset{?}{\simeq} D^a_\square(X_S, \mathcal{O}^+/\pi)$.

("mod p Riemann-Hilbert")

Sketch of proof: Using results of Fargues-Scholze, reduces to computing $\underline{R\mathrm{Hom}}_{D(S_v,E)}(?,??)$ for

$$?,?? \in \{\underbrace{\underline{E}}_{=R\Gamma_* \mathcal{O}_{X_S}},\ \underbrace{\mathcal{G}_{a,S^\#}}_{=R\Gamma_*(i_{S^\#*}\mathcal{O}_{S^\#})}\}.$$

$S^\#$ fixed untilt of S over E.

Easy when $? = \underline{E}$. When $E = \mathcal{G}_{a,S^\#}$,

$$\underline{R\mathrm{Hom}}_{D(S_v,\underline{E})}(\mathcal{G}_{a,S^\#},\underline{E}) \cong \mathcal{G}_{a,S^\#}(-1)[1].$$

$$\underline{R\mathrm{Hom}}_{D(S_v,\underline{E})}(\mathcal{G}_{a,S^\#},\mathcal{G}_{a,S^\#}) \simeq \mathcal{G}_{a,S^\#} \oplus \mathcal{G}_{a,S^\#}(-1)[1].$$

Using the short exact sequences

$$0 \to \underline{E} \to \mathbb{B}_{\inf} \xrightarrow{\varphi-\mathrm{id}} \mathbb{B}_{\inf} \to 0,$$

$$0 \to \mathbb{B}_{\inf} \xrightarrow{\xi} \mathbb{B}_{\inf} \to \mathcal{G}_{a,S^\#} \to 0,$$

with $\mathbb{B}_{\inf} = \mathbb{A}_{\inf}[\frac{1}{\pi}]$, $\mathbb{A}_{\inf} = W_{\mathcal{O}_E}(\mathcal{O}^+)$,

reduces to showing $\underline{R\mathrm{Hom}}_{D(S_v,\underline{\mathcal{O}_E})}(\mathbb{A}_{\inf},\mathbb{A}_{\inf})$

$$\overset{\sim}{=}_a \mathbb{A}_{\inf}\langle F^{\pm 1}\rangle^{\wedge}_{(\pi,[\varpi])}$$

This in turn is deduced (by π-completeness) $(\pi,[\varpi])$-adic completion.
from $R\underline{\mathrm{Hom}}_{D(S_v, \mathbb{F}_q)}(\mathcal{O}^+, \mathcal{O}^+) \simeq_a \mathcal{O}^+\langle F^{\pm 1}\rangle_{([\varpi])}$.

This is proved using Breen's computation:

$$R\underline{\mathrm{Hom}}_{D(\underset{\substack{\uparrow \\ \text{alg. perfect v-site}}}{\mathrm{Spec}(\mathbb{F}_q)}, \mathbb{F}_q)}(\mathbb{G}_a, \mathbb{G}_a) \simeq \mathbb{G}_a[F^{\pm 1}].$$

■

<u>Rks</u>: a) From Th 2, one deduces for each $M \in \mathrm{Perf}(X_S)$ a natural isomorphism

$$R\tau_*(R\underline{\mathrm{Hom}}_{\mathcal{O}_{X_S}}(M, \mathcal{O})) \simeq R\underline{\mathrm{Hom}}_{D(S_v, \underline{E})}(R\tau_* M, \underline{E})$$

"relative Serre duality" on the FF-curve.

b) Original motivation for Th 2 comes from defining and studying an l-adic Fourier transform ($l \neq p$) for Banach-Colmez spaces, relevant to the study of LLC in the context of Fargues Scholze's geometrization program.

(Again, very analogous to unipotent perfect group schemes!)

Call $\mathcal{BC}(S)$ the essential image of $\mathrm{Perf}(X_S)$ in $D(S_v, \underline{E})$ by $R\tau_*$. Th 2 and its proof show:

- $\mathcal{BC}(S)$ is the full subcategory of $D(S_v, \underline{E})$ of objects which v-locally on S belong to the smallest idempotent complete stable ∞-subcategory spanned by $\underline{E}$ and $\mathbb{G}_{a,S^\#}$ ($S^\#$ untilt of S/E).

$\Rightarrow$ When $E = \mathbb{Q}_p$ and $S = \mathrm{Spa}(C, C^+)$, $\mathcal{BC}$ agree with the bounded derived cat. of BC-spaces.

- The functor $\mathbb{D} := R\underline{\mathrm{Hom}}_{D(S_v, \underline{E})}(-, \underline{E})$ defines an auto-duality of $\mathcal{BC}$.

  E.g.: $\mathbb{D}(\underline{E}) = \underline{E}$, $\mathbb{D}(\mathbb{G}_{a,S^\#}) = \mathbb{G}_a^\#(-1)[-1]$.

Moreover, by Th 1, can make sense of $\mathcal{BC}(S)$ for any (small) v-stack S, in particular for any rigid space.

Ex: 1) $\mathcal{BC}(\mathrm{Spd}\,\bar{k}) = D^b$(isocrystals over k)

(Anschütz)

2) $\mathcal{BC}(\mathrm{Div}^1_K) \sim (\varphi,\Gamma)$-modules over the Robba ring

/

W_E-equivariant perfect complexes on $X_{\hat{\bar{K}},E}$

[analogue of $\bar{\mathbb{Q}}_\ell$-lisse sheaves on Div^1_K = ℓ-adic reps of W_K, $\ell \neq p$)

Q: Can one make this category more explicit for S smooth rigid space?

(Contains $\underline{E}$-local systems (for the v-top) and v-vector bundles ((....) Higgs bundles))

Q: Let $f : S' \to S$ smooth proper morphism of rigid spaces. Is $\mathcal{BC}$ stable under Rf_*? Do we have a projection formula?

When $\exists$ "integral structure", can be deduced from work of Mann.

Here is a non-trivial example outside this case.

Ex: (Scholze's Jacquet-Langlands functor applied to an algebraic representation.) $E = \mathbb{Q}_p$, $S' = \mathbb{P}^1_C \to \mathrm{Spd}(C) = S$.
$C/\mathbb{Q}_p$.

Let $\mathbb{L}$ $\mathbb{Q}_p$-local system on $\mathbb{P}^1_C$ obtained by descending the trivial rank 2 local system $\underline{\mathbb{Q}_p^2}$ on $\mathcal{M}_{LT,\infty,C}$ along $\mathcal{M}_{LT,\infty,C} \to \mathbb{P}^1_C$.

$\uparrow$ with standard action of $GL_2(\mathbb{Q}_p)$

$\uparrow$ $GL_2(\mathbb{Q}_p)$-torsor

Using the s.e.s.

$$0 \to \mathbb{L} \to BC_{\mathbb{P}^1_C}(\mathcal{O}(\tfrac{1}{2})) \to \overset{\hat{\mathcal{O}}}{\overset{\shortparallel}{\mathbb{G}_{a,\mathbb{P}^1_C}}} \to 0,$$

(Faltings's crystalline-étale comparison for Tate module universal p-div gp over $\mathcal{M}_{LT,\infty}$), can compute

$$H^i(\mathbb{P}^1_C, \mathbb{L}) = \begin{cases} BC(\mathcal{O}(-1/2)) & i=1 \\ BC(\mathcal{O}(1/2)) & i=2 \\ 0 & \text{otherwise} \end{cases}$$

not finite dim'l $\mathbb{Q}_p$-vs, but still Banach-Colmez spaces.

Also, note that $\mathbb{L} \underset{\mathbb{Q}_p}{\otimes} \mathbb{G}_a = \hat{\mathcal{O}}(-1) \oplus \hat{\mathcal{O}}(1)$ hence $R\nu_*(-) = \mathcal{O}(1) \oplus \mathcal{O}(-3)[-1]$. Thus

$$R\Gamma(\mathbb{P}^1_C, \mathbb{L} \underset{\mathbb{Q}_p}{\otimes} \mathbb{G}_a) = \begin{cases} H^0(\mathbb{P}^1_C, \mathcal{O}(1)) = C^2 & i=0 \\ H^1(\mathbb{P}^1_C, \mathcal{O}(-3)) \simeq C^2 & i=2 \\ 0 & \text{otherwise.} \end{cases}$$

so projection formula holds if we $\otimes$ transported from the FF-curve!

category more explicit for S smooth rigid space. It contains $\underline{E}$-local systems (for the v-topology) and v-vector bundles ($\longleftarrow\cdots\longrightarrow$ Higgs bundles).

This category won't be stable by various operations in general, e.g pushforward. An even more ambitious goal would be to show that

$\mathrm{Perf}_{\mathbb{F}_q} \ni S \longmapsto D^{\blacksquare}_{qcoh}(X_S)$ satisfies v-descent

allowing to define for any small v-stack an ∞-cat

$\mathcal{BC}_{\blacksquare}(S)$ + 6-functor formalism.

($\mathcal{BC}(S)$ would then be recovered as the subcat. of dualizable object.)