Six functor formalism of analytic stacks and examples

Adam Dauser

May 24, 2025

1 Talk 3: Six functor formalism of analytic stacks

1.1 Introduction to six functor formalisms

The first six functor formalism was constructed by Grothendieck et al. [SGA4] to handle (relative) étale cohomology (with compact supports). Let's briefly recall how it goes (with coefficients in $\Lambda = \mathbb{Z}/n\mathbb{Z}$, $\Lambda = \mathbb{Z}_\ell$ or $\Lambda = \mathbb{Q}_\ell$, where n resp. ℓ is tacitly assumed to be a unit on all schemes):

1. To a scheme X, associate the étale site $X_{\text{ét}}$ consisting of étale maps $f : Y \to X$ with covers given by jointly surjective maps. Form the subcategory $\mathcal{D}(X_{\text{ét}})$ of the derived category[1] of sheaves of Λ-modules on X whose cohomology sheaves are constructible;

2. $\otimes^{\mathrm{L}}_\Lambda$ equips $\mathcal{D}(X_{\text{ét}})$ with the structure of a symmetric monoidal category, and we can form $\mathrm{R}\underline{\mathrm{Hom}}$ as its right-adjoint;

3. pulling back sheaves along a map $f : X \to Y$ yields $f^* : \mathcal{D}(Y_{\text{ét}}) \to \mathcal{D}(X_{\text{ét}})$ with right adjoint, the *relative étale cohomology "push forward" functor* $\mathrm{R}f_*$;

4. if $j : U \to X$ is an open immersion, j^* has a left adjoint, the *extension by zero functor* $j_!$. Factoring a separated map of finite type $f : X \to Y$ as

$$X \overset{j}{\hookrightarrow} \overline{Y} \xrightarrow{\overline{f}} Y$$

an open immersion followed by a proper morphism, we define the *relative étale cohomology with compact support "exceptional push forward" functor* $\mathrm{R}f_! = \mathrm{R}f_* j_!$, which admits a right adjoint $\mathrm{R}f^!$. These are the six functors.

5. we have the projection formula: Given a separated map of finite type $f : X \to Y$, there is a "canonical" isomorphism $\mathrm{R}f_!(A \otimes^{\mathrm{L}}_\Lambda f^* B) \simeq (\mathrm{R}f_! A) \otimes^{\mathrm{L}}_\Lambda B$ functorial in A and B;

6. and proper base change: Given a fibre square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{f} & Y, \end{array}$$

there is a "canonical" isomorphism $g^* \mathrm{R}f_!(A) \simeq \mathrm{R}f'_! g'^{,*}(A)$ functorial in A.

[1] Starting here, we will treat all derived categories as ∞-categories.

Remark 1. *The meaning of the word* canonical *here is a bit subtle and all the compatibility conditions between these natural isomorphisms are quite involved. This problem is worse yet for analytic stacks. Yifeng Liu and Weizhe Zheng give a precise formulation of all this data, which Lucas Mann nicely repackaged as a lax symmetric monoidal functor*

$$\mathrm{Corr}(\mathrm{Sch}, E) \to \mathrm{Cat}_\infty,$$

where $\mathrm{Corr}(\mathrm{Sch}, E)$ *is a certain* ∞*-category, whose objects are schemes with maps given by correspondences*

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{g} & & \searrow^{f} & \\ X & & & & Y \end{array}$$

with f *separated of finite type, and composition given by taking fibre products. In general, the class* E *comprises* !-able maps *i.e. those we for which we can form* $f_!$.

As notation like $\mathrm{R}f^!$ is a bit of a red herring anyway, we will drop R's and L's from the notation.

Everything can (almost) be translated to quasi-coherent sheaves on affine schemes:

1. To any ring R associate $\mathcal{D}(\mathrm{Spec}(R)) \coloneqq \mathcal{D}\mathrm{Mod}(R)$;

2. we have $\otimes_R^{\mathrm{L}}$ and RHom_R;

3. for $f : \mathrm{Spec}(R) \to \mathrm{Spec}(S)$, we have $f^* \coloneqq - \otimes_S^{\mathrm{L}} R : \mathcal{D}(\mathrm{Spec}(S)) \to \mathcal{D}(\mathrm{Spec}(R))$ with right adjoint $(-)_S = f_*$;

4. we set $f_! \coloneqq f_*$, which admits a right adjoint $f^! = \mathrm{RHom}_R(S, -)$;[2]

5. there is a canonical isomorphism $f_!(A \otimes f^* B) \simeq (f_! A) \otimes B$ functorial in A and B;

6. given a fibre square

$$\begin{array}{ccc} \mathrm{Spec}(S' \otimes_S R) & \xrightarrow{f'} & \mathrm{Spec}(R) \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ \mathrm{Spec}(S') & \xrightarrow{f} & \mathrm{Spec}(S), \end{array}$$

   such that g is *flat*, there is a canonical isomorphism $g^* f_!(A) \simeq f'_! g'^{,*}(A)$ functorial in $A \in \mathcal{D}(Y')$.

Remark 2. *One can eliminate the condition that* g *is flat by working with animated rings*[3] *instead—this means we would be taking* $\mathrm{Spec}(S' \otimes_S^{\mathrm{L}} R)$ *instead. Then we obtain a quasi-coherent six functor formalism*

$$\mathcal{D} : \mathrm{Corr}(\mathrm{AniSch}_{\mathrm{aff}}) \to \mathrm{Cat}_\infty.$$

In the following we will also secretly work with animated analytic rings/stacks.

[2] Not to be confused with the functor from Grothendieck duality.

[3] These can e.g. modelled via simplicial rings

1.2 Six functors of affine analytic stacks

Definition 3. *Any map $f : R \to S$ of analytic rings can be factored into*

1. *a* proper *map* $(R^{\triangleright}, \mathrm{Mod}_R) \to (S^{\triangleright}, \{M \in \mathrm{Mod}_{S^{\triangleright}} | M \otimes_{S^{\triangleright}} R^{\triangleright} \in \mathrm{Mod}_R\})$*, the latter is called the* induced analytic ring structure*;*[4]

2. *an* open immersion $((S^{\triangleright}, \{M \in \mathrm{Mod}_{S^{\triangleright}} | M \otimes_{S^{\triangleright}} R^{\triangleright} \in \mathrm{Mod}_R\})) \to (S^{\triangleright}, \mathrm{Mod}_S)$.[5]

Remark 4. *To understand what open immersions can look like, think back to the example of étale sheaves. If $j : U \hookrightarrow X$ is an open immersion, $j_!\Lambda \in \mathcal{D}(X_{\text{ét}})$ is an idempotent co-algebra: There is a co-unit $\epsilon : j_!\Lambda \to \Lambda$ and co-multiplication $c : j_!\Lambda \xrightarrow{\simeq} j_!\Lambda \otimes j_!\Lambda$. $\mathcal{D}(U_{\text{ét}})$ is exactly the category of co-modules over this co-algebra i.e. sheaves such that $\mathcal{F} \xrightarrow{\simeq} \mathcal{F} \otimes j_!\Lambda$. If the complement*[6] *is $i : Z \to X$, $i_*\Lambda$ is similarly an idempotent algebra and we can recover $j_!\Lambda$ as the fibre*

$$j_!\Lambda \to \Lambda \to i_*\Lambda.$$

The same is true for analytic rings: Open immersions of analytic stacks come from "complementary" idempotent algebras $A \in \mathrm{Mod}_R$.

Remark 5. ***Warning:*** *This means that any map of analytic rings with discrete analytic ring structure is proper. In particular, these words do not match up with the nomenclature of algebraic geometry.*

To combine what we have learnt from these two examples,

1. To an analtyic ring $R = (R^{\triangleright}, \mathrm{Mod}_R)$, associate $\mathcal{D}(\mathrm{AnSpec}(R)) \coloneqq \mathcal{D}(\mathrm{Mod}_R)$;

2. we have $\otimes \coloneqq \otimes_R^{\mathrm{L}}$ with right adjoint RHom_R;

3. for $f : \mathrm{AnSpec}(R) \to \mathrm{AnSpec}(S)$, we have $f^* \coloneqq - \otimes_S^{\mathrm{L}} R : \mathcal{D}(\mathrm{Spec}(S)) \to \mathcal{D}(\mathrm{Spec}(R))$ with right adjoint $(-)_S = f_*$;

4. factor a map f into a proper morphism $\overline{f}$ and open immersion j. If j^* admits a left adjoint satisfying the projection formula, we call f !-able.[7] In that case set $f_! = \overline{f}_* j_!$, which admits a right adjoint $f^!$. Explicitly $f^! = \mathrm{RHom}_R(S, -)$ for proper maps and j^* for open immersions;

5. there is a canonical isomorphism $f_!(A \otimes f^*B) \simeq (f_!A) \otimes B$ functorial in A and B;

6. given a fibre square

$$\begin{array}{ccc} \mathrm{Spec}(S' \otimes_S^{\mathrm{L}} R) & \xrightarrow{f'} & \mathrm{Spec}(R) \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ \mathrm{Spec}(S') & \xrightarrow{f} & \mathrm{Spec}(S), \end{array}$$

   there is a canonical isomorphism $g^* f_!(A) \simeq f'_! g'^{,*}(A)$ functorial in $A \in \mathcal{D}(Y')$.

[4] Of course, you have to complete $S^{\triangleright}$ w.r.t. this pre-analytic ring structure.

[5] This will only match the cohomological definition of "open immersion" if j^* admits a left adjoint $j_!$ satisfying the projection formula. Examples coming from complementary idempotent algebras as in the following remark are of this form.

[6] With any scheme structure e.g. the reduced one.

[7] E.g. any proper map and any open immersion given by solidifying finitely many elements is !-able.

1.3 Supplement: General notions in 6 functor formalisms

Definition 6. *(Poincaré duality.) Let $\mathcal{D} : Corr(\mathcal{C}, E)^{\otimes} \to Cat_{\infty}$ be a 6-functor formalism, $f : X \to Y$ a map in E, then we call:*

1. *$K \in \mathcal{D}(X)$ f-smooth, if for the f-smooth dual $\mathbb{D}_f^{sm}(K) := \mathrm{Hom}(K, f^! 1_Y)$ the canonical map*
$$\pi_1^* K \otimes \pi_2^* \mathbb{D}_f^{sm}(K) \to \mathrm{Hom}(\pi_2^* K, \pi_1^! K)$$
is an isomorphism. In particular, if $K = 1_X$ is f-smooth, $f^! \simeq f^*(- \otimes \mathbb{D}_f^{sm}(1_X))$;

2. *f cohomologically smooth, if $1_Y \in \mathcal{D}(Y)$ is f-smooth and the dualising sheaf $\mathbb{D}_f^{sm}(1_Y)$ is $\otimes$-invertible.*

3. *$L \in \mathcal{D}(Y)$ f-proper, if for the f-proper dual $\mathbb{D}_f^{prop}(L) := \pi_{2,*}\mathrm{Hom}(\pi_1^* L, \Delta_!(1_X))$ the canonical map*
$$f_!(L \otimes \mathbb{D}_f^{prop}(L)) \to f_* \mathrm{Hom}(L, L)$$
is an isomorphism.

Proposition 7. *f-smooth resp. f-proper objects and f-smooth maps are stable under base change and satisfy !-descent.*

Moreover, f-proper resp. f-smooth objects are stable under cones and shifts. The f-proper resp. f-duals can be computed by applying the exact functors $\mathbb{D}_f^{sm}$ resp. $\mathbb{D}_f^{prop}$.

Further, we have a diagram
$$X \xrightarrow{f} Y \xrightarrow{g} Z, \qquad h = g \circ f : X \to Z$$
where f is cohomologically smooth with dualising sheaf L and $K \in \mathcal{D}(X)$ is h-proper with h-proper dual K', then $f_! K$ is g-proper with g-proper dual $f_!(L \otimes K')$.[8]

Proof. All of these but the last can be found in the Notes to Six functor formalisms. We sketch the last:
$$\begin{aligned} g_!(f_! K \otimes -) &\simeq g_! f_!(K \otimes f^* -) \simeq h_!(K \otimes f^* -) \\ &\simeq h_* \mathrm{Hom}(K', f^* -) \simeq g_* f_* \mathrm{Hom}(K' \otimes L, f^! -) \\ &\simeq g_* \mathrm{Hom}(f_!(K' \otimes L), -) \end{aligned}$$
Using the $(\infty, 2)$-category $LZ_{\mathcal{D}}$, it is not hard to make this into a formal argument. □

1.4 Passing to stacks

"Stacks" might sound a bit scary at first, but as will treat them analogously to the natural continuation of the tower of generalisations

affine schemes → separated qc schemes → schemes

given by the following: Separated (qc) schemes X are exactly those sheaves in the Zariski topology[9], that admit a (finite) Zariski cover $\widetilde{X} \to X$ by an affine scheme and such that the map $X \to X \times X$ is affine[10]. Similarly, schemes are those Zariski sheaves admitting a Zariski cover $\widetilde{X} \to X$ by a separated qc scheme and such that $\Delta : X \to X \times X$ is quasi-compact and separated.

[8] Of course, there is a dual statement as well.

[9] we are working with the big Zariski site on affine schemes here.

[10] For a general Zariski sheaf X, we say $\Delta : X \to X \times X$ is affine, if for any affine scheme Y and map $Y \to X \times X$, the fibre product $Y \times_{Y \times Y} X \to Y$ is affine.

Exchanging the Zariski for the étale topology, this would give us

$$\begin{aligned} \text{affine schemes} &\to \text{qc algebraic spaces with affine diagonal} \\ &\to \text{algebraic spaces} \to \text{Deligne-Mumford stacks} \to \cdots \end{aligned}$$

(the list now proceeds infinitely to the right if we allow sheaves of 2-groupoids, ..., anima instead).

The technically most important fact about these topologies is that $\mathcal{D}(-)$ satisfies descent for them.

In the world of analytic stacks, there is just no relevant distinguished class of "analytic schemes", because we will choose a far far finer topology called the !-topology, which roughly is defined to be the finest manageable topology for which $\mathcal{D}(-)$ satisfies descent.

Definition 8. *A proper map $f : A \to B$ of analytic rings is called* descendable*, if*

$$A \in \langle f_* M | M \in \mathcal{D}(B) \rangle_{\text{fin. lim., retracts}}.$$

We further call a map descendable of index $\leq m$*, if for $F = \text{fib}(A \to B)$, $F^{\otimes_A m} \to A$ is null-homotopic.*

Example 9. *A split epimorphism is descendable of index ≤ 1.*

Remark 10. *Zariski covers for open immersions of analytic stacks and descendable proper maps are examples of !-cover.*

Example 11. *Faithfully flat maps of countable presentation are an example of descendable maps. Notably, not all faithfully flat maps will do, because we will later need to impose some compatibility with $f_!, f^!$. More generally:*

Proposition 12. *Let $g_{i<j} : A_i \to A_j$ be a diagram of shape $(\mathbb{N}, \leq)$ in the category of analytic rings and let $(f_i : A \to A_i)_{i \in \mathbb{N}}$ be a cocone consisting of proper maps of analytic rings, that are descendable of index $\leq m$, then*

$$A \to \varinjlim A_i$$

is descendable.

As an enlightening exercise, let us first run this programme for the Zariski topology and restricting to analytic stacks with discrete analytic ring structure and let's extend to quasi-compact and separated schemes.

1. define $\mathcal{D}(X) = \varprojlim_i \mathcal{D}(U_i)$ for any open affine cover U_i of a quasi-compact separated scheme X (such that $\bigsqcup_i U_i \to X$ is affine in our case);

2. this automatically becomes equipped with $\otimes$ and Hom;

3. given $f : Y \to X$, take a finite open cover $\{U_i\}$ of X and a compatible one $\{V_{ij}\}$ on Y. Then we can glue the maps $\mathcal{D}(U_i) \to \mathcal{D}(V_{ij})$. This is independent of the covers by Zariski descent. We obtain f_* as its right adjoint;

4. defining $f_!$ is a bit more subtle now. We want proper base-change to hold. If we ponder this for fibre products along $j : U \to X$ with fibre $V \to Y$, a definition is forced on us if $f : Y \to X$ is affine: Take an open affine cover U_i of X, we obtain $f_i : f^{-1}(U_i) \to U_i$ and define $f_!$ via descent from $f_{i,*} = f_{i,!}$. This will automatically satisfy proper base change and the projection formula.[11]

[11] ~[Man23, Proposition A.5.12]

For any map $f : Y \to X$, there exists a open affine cover $j_i : V_i \to Y$ such that $g : \bigsqcup_i V_i \to Y$ and $f \circ g : \bigsqcup_i V_i \to X$ are affine.[12] Set n-fold self intersections $g_n : \bigsqcup_{(i_1,\ldots,i_n)} V_{i_1} \cap \cdots \cap V_{i_n} \to Y$. Define

$$f_! := \varinjlim_{n \in \Delta} (f \circ g_n)_! g_n^! .$$ [13]

Again, this admits a right adjoint.

The general mechanism, that we will apply to the six functor formalism of affine algebraic stacks is:

Proposition 13 (Programme DESCENT)**.** *Denote by* AnStack *the category of* !*-sheaves*[14] *a.k.a. analytic stacks. Define*

1. *for any analytic stack $\mathcal{X}$, set $\mathcal{D}(\mathcal{X}) = \varprojlim_{(X \to \mathcal{X})} \mathcal{D}(X)$, where the inverse limit is running over X affine (think a quasi-coherent sheaf on $\mathcal{X}$ can be pulled back to X);*
2. *$\otimes$ and* Hom *are automatic;*
3. *f^* and f_* by descent;*
4. *we can define $f_!$ for a slightly mysterious class of* !*-able maps. For affine maps of analytic stacks, we may define it via descent from $f_!$'s for affine maps. Then we can induct.*

Fur a full construction vis Scholze's lecture notes:[15]

Remark 14. *We genuinely need* Mod_R *to be R-modules on solid abelian groups, even if we only care about discrete modules in the end, these are not preserved by $f_!$ for e.g. open immersions of schemes, that are not quasi-compact.*

2 Talk 5: Examples of analytic stacks

2.1 Betti stacks

Consider the functor

$$\mathrm{ProFin}^{\mathrm{light}} \to \mathrm{AnStack}$$

$$S \mapsto \mathrm{AnSpec}(\mathrm{Cont}(S, \mathbb{Z})).$$

For a surjective map of light profinite sets, $S \to T$, the induced map $Cont(T, \mathbb{Z}) \to Cont(S, \mathbb{Z})$ can be written as an $\mathbb{N}$-indexed colimit of split surjective maps and is thus descendable. Hence the functor takes hypercovers to !-hypercovers and extends to a unique colimit preserving functor

$$-_{\mathrm{Betti}} : \mathrm{CondAni}^{\mathrm{light}} \to \mathrm{AnStack}.$$

This is controllable, if we plug in a finite dimensional locally compact Hausdorff space:

Proposition 15. *Let S be a finite dimensional compact Hausdorff space and $f : S' \to S$ a surjection from a light profinite set. Then $f_* \mathbb{Z} \in \mathcal{D}(S, \mathbb{Z})$ is descendable.*

[12]Indeed, find an open affine cover $U_i \to X$ such that $\bigsqcup_i U_i \to X$ is affine, pulling back yields that $\bigsqcup_i f^{-1}(U_i) \to Y$ is affine. Now we may find finite open affine covers $V_{ij} \to f^{-1}(U_i)$ of the quasi-compact spaces $f^{-1}(U_i)$.

[14]Often a slightly technical notion between sheaf and hypersheaf is assumed instead.

[15]https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf

Corollary 16. *For any analytic stack X and any finite-dimensional compact Hausdorff space S, one has a natural equivalence*

$$\mathcal{D}(X \times S_{\text{Betti}}) \simeq \mathcal{D}(S, \mathcal{D}(X)),$$

where the later denotes the ∞-category of sheaves on S with values in $\mathcal{D}(X)$.

Proof. If S is a light profinite set, the global sections functor $\mathcal{D}(S, \mathcal{D}(X)) \to Mod_{\text{Cont}(S,\mathbb{Z})}\mathcal{D}(X) \simeq \mathcal{D}(X \times \text{AnSpecCont}(S, \mathbb{Z}))$ is an equivalence. In general, choose a surjection $f_0 : S_0 \to S$, with Čech nerve $f_\bullet : S_\bullet \to S$. On each step, we get

$$\mathcal{D}(X \times S_{n,\text{Betti}}) \simeq \mathcal{D}(S_n, \mathcal{D}(X)) \simeq Mod_{f_{n,*}\mathbb{Z}}(\mathcal{D}(S, \mathcal{D}(X)))$$

with the second equivalence induced by $f_{n,*}$. This is functorial in n and we get equivalences

$$\mathcal{D}(X \times S_{\text{Betti}}) \simeq \varprojlim_{\Delta} \mathcal{D}(X \times S_{\bullet,\text{Betti}}) \simeq \varprojlim_{\Delta} Mod_{f_{n,*}\mathbb{Z}}(\mathcal{D}(S, \mathcal{D}(X)))$$

where the first equivalence comes from the fact that $S_{\bullet\text{Betti}} \to S$ is a !-hypercover. Finally, the previous lemma, that $\mathbb{Z} \to f_*\mathbb{Z}$ is descendable implies that

$$\varprojlim_{\Delta} Mod_{f_{n,*}\mathbb{Z}}(\mathcal{D}(S, \mathcal{D}(X))) \simeq \mathcal{D}(S, \mathcal{D}(X)).$$

□

Theorem 17. *(Tannakian reconstruction.) Let S be a finite-dimensional compact Hausdorff space. For any analytic stack X, maps $X \to S_{\text{Betti}}$ are equivalent to $\mathcal{D}(\mathbb{Z})$-linear colimit preserving symmetric monoidal functors*

$$\mathcal{D}(S, \mathbb{Z}) \to \mathcal{D}(X)$$

such that there exists some !-cover $X'_i \to X$ by affine analytic stacks, for which the composite functor

$$\mathcal{D}(S, \mathbb{Z}) \to \mathcal{D}(X) \to \mathcal{D}(X'_i)$$

preserves connective objects.

Moreover, such functors are equivalently given by collections of idempotent algebras $A_Z \in \mathcal{D}(X)$ for every closed subset $Z \subseteq S$, such that $Z \mapsto A_Z$ sends limits to colimits and finite colimits to limits [16]*, such that $A_Z|_{X'_i} \in \mathcal{D}(X'_i)$ are connective.*

Remark 18. *By abuse of notation, we will also write S_{Betti} for the base change to $\text{AnSpec}(\mathbb{C}_{\text{gas}})$.*

Lemma 19. *Let X be a complex analytic space, then $X(\mathbb{C})_{\text{Betti}}$ admits a !-cover consisting of maps*

$$\text{AnSpec}(\text{Cont}(S, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}_{\text{gas}}) \to X(\mathbb{C})_{\text{Betti}}$$

induced by maps $S \to X$, where $S = \varprojlim_{n \in \mathbb{N}} S_n$ is a light profinite set such that for every $s_n \in S_n$, $\text{im}(S \times_{S_n} \{s_n\} \to X(\mathbb{C}))$ is compact Stein.

Proof. By covering $X(\mathbb{C})$ with compact Stein subsets, we may assume X is compact Stein. Set $S_0 = \{0\}$ and $X_0 = X$. Now inductively cover X_i for $i \in S_n$ with Stein subsets X_j indexed by a finite set $j \in I_i$ such that in each set X_j, there exists a point $x_j \in X_j$ that is not contained in any other $X_{j'}$ for $j' \in I_i$. Set $S_{n+1} = \sqcup_{i \in S_n} I_i \to S_n$. Under these conditions, we see that for

[16] I.e. is a morphism of locals between the locale opposite to the frame of closed subsets and the locale of idempotent algebras.

any $s \in S := \varprojlim_{n\in\mathbb{N}} S_n$, $\bigcap_{n\in\mathbb{N}} X_{s_n} =: \{f(s)\}$ is a single point. One readily checks, that the map $f : S \to X(\mathbb{C})$ is surjective, continuous; and that for every $s_n \in S_n$,

$$\mathrm{im}(S \times_{S_n} \{s_n\} \to X(\mathbb{C})) = X_{s_n}.$$

As $-_{\mathrm{Betti}}$ sends covers to !-covers, the associated map

$$S_{\mathrm{Betti}} = \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}_{\mathrm{gas}}) \to X(\mathbb{C})_{\mathrm{Betti}}$$

is a !-cover. □

Lemma 20. *Let X be a complex-analytic space. Then $\mathcal{O}(Z)^\dagger$ are idempotent and satisfy the conditions of the Tannakian reconstruction theorem for $X(\mathbb{C})$. For any compact Stein subset $Z \subseteq X$, $X \times_{X(\mathbb{C})_{\mathrm{Betti}}} Z(\mathbb{C})_{\mathrm{Betti}} = \mathrm{AnSpec}(\mathcal{O}(Z)^\dagger)$.*

Proof. That $\mathcal{O}(Z)^\dagger$ satisfy the appropriate conditions is proved in [complex.pdf].[17] The second part is a direct consequence of the proof of Tannakian reconstruction. □

Theorem 21. *Let X be a complex-analytic space, then $X \to X(\mathbb{C})_{\mathrm{Betti}}$ is an epimorphism of analytic stacks.*

Proof. We may check that $X \to X(\mathbb{C})_{\mathrm{Betti}}$ is an epimorphism of analytic stacks on a !-cover consisting of

$$\mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}_{\mathrm{gas}}) \to X(\mathbb{C})_{\mathrm{Betti}}$$

of the form of the previous lemma. This means, we need to show

$$X \times_{X(\mathbb{C})_{\mathrm{Betti}}} \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}_{\mathrm{gas}})) \to \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}_{\mathrm{gas}}))$$

is a !-cover. Setting,

$$X_n = \bigsqcup_{s_n \in S_n} \mathrm{im}(S \times_{S_n} \{s_n\} \to X(\mathbb{C})),$$

we use the previous lemma to see that this fiber product is affine and can be computed as

$$\mathrm{AnSpec}(\varinjlim_n \mathcal{O}(X_n)^\dagger) \to \mathrm{AnSpec}(\varinjlim_n \mathrm{Cont}(S_n,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}_{\mathrm{gas}})).$$

As a proper map of affine analytic stacks, it suffices that the corresponding map of analytic rings is descendable. But for each $n \in \mathbb{N}$,

$$\mathbb{C}_{\mathrm{gas}}{}^{S_n} = \mathrm{Cont}(S_n,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}_{\mathrm{gas}} \to \mathcal{O}(X_n)^\dagger$$

is split and thus descendable of index ≤ 1. Thus their colimit is descendable. □

2.2 Algebraic model for Fourier analysis

Definition 22. *For every closed subset $I \subseteq S^1$, set $K_I := \{z \in \mathbb{C}\backslash\{0\} | \arg(z) \in I\} \subseteq \mathbb{C}\backslash\{0\}$. Define*

$$\mathcal{O}^{\dagger,\mathrm{alg}}(K_I) = \{f \in \mathcal{O}^\dagger(K_I) | \exists N \geq 0 : |f(z)| \leq |z|^N \text{ as } |z| \to \infty, |f(z)| \leq |z|^{-N} \text{ as } |z| \to 0\}.$$

These are idempotent algebras compatible with intersections and finite unions and we have $\mathcal{O}^{\dagger,\mathrm{alg}}(S^1) = \mathbb{C}[z^{\pm 1}]$. By Tannakian reconstruction, this yields a map

$$\arg : \mathbb{G}^{\mathrm{alg}}_{m,\mathbb{C}_{\mathrm{gas}}} \to (S^1)_{\mathrm{Betti}}.$$

[17] This reference works with $\mathbb{C}^{p\text{-liq}}$ instead of $\mathbb{C}^{\mathrm{gas}}$ but carefully inspecting all arguments involved

Definition 23. *Define* $\widetilde{\mathbb{G}}_m$ *as the fibre product (of abelian analytic group stacks) of*

$$\begin{array}{ccc} \widetilde{\mathbb{G}}_m & \longrightarrow & \mathbb{G}_m \\ \downarrow & & \downarrow \text{arg} \\ \mathbb{R}_{\text{Betti}} & \longrightarrow & (S^1)_{\text{Betti}} \end{array}$$

We automatically have a short exact sequence:

$$0 \to \mathbb{Z} \to \widetilde{\mathbb{G}}_m \to \mathbb{G}_m \to 0$$

We have a map $\log : \widetilde{\mathbb{G}}_m \to \mathbb{A}^1$ *obtained by glueing together* $\log \in \mathcal{O}^{\dagger,\text{alg}}(K)$ *defined on sectors (note it satisfies the growth condition). This is interesting as it is possible to define the Fourier transform on* $\mathrm{R}\Gamma(\widetilde{G}_m)$*.*

2.3 Analytic deRahm stacks

The classic Riemann-Hilbert correspondence is between vector bundles with a flat connection and their associated local systems of flat sections. Such sections fulfil a special kind of partial differential equation. This correspondence has been generalised to so called (regular holonomic) D-modules, which are (sheaves of) modules over an (non-commutative sheaf of) algebra(s) of differential operators, which in turn correspond to constructible sheaves.

If we impose stronger analytic properties on our D-modules, we will in fact get more than a mere equivalence of certain subcategories of sheaves. We will get an equivalence of analytic stacks!

Definition 24. *Let* X *be a complex manifold, define* $\widehat{\Delta(X)} \subset X \times X$ *to be the union of all infinitesimal thickenings of the diagonal*[18]*. Define the* algebraic deRahm stack *as* $X^{\text{dR}} = X/\widehat{\Delta(X)}$*.*

Define $\Delta(X)^\dagger = \varprojlim_{U \subset \Delta(X)}$ *as union of all open subsets of* $X(\mathbb{C})$ *the* overconvergent neighborhood of the diagonal*. Define the* analytic deRahm stack $X^{an}_{\text{dR}} = X/\Delta(X)^\dagger$*.*

Next we will see how these relate to the aforementioned D-modules.

2.4 Cartier duality

As is so often the case, one can reduce the discussion to the case that X is an affine line. The algebraic deRahm stack becomes $\mathbb{G}^{\text{dR}}_{a,\mathbb{C}_{\text{gas}}} = \mathbb{G}_{a,\mathbb{C}_{\text{gas}}}/\widehat{\mathbb{G}_{a,\mathbb{C}_{\text{gas}}}}$ and the analytic one $\mathbb{G}^{an,\text{dR}}_{a,\mathbb{C}_{\text{gas}}} = \mathbb{G}^{an}_{a,\mathbb{C}_{\text{gas}}}/\mathbb{G}^\dagger_{a,\mathbb{C}_{\text{gas}}}$, where $\mathbb{G}^\dagger_{a,\mathbb{C}_{\text{gas}}} := (\{0\} \subset \mathbb{G}^{an}_{a,\mathbb{C}_{\text{gas}}})^\dagger$. We want to write $\mathcal{D}(\mathbb{G}^{\text{dR}}_{a,\mathbb{C}_{\text{gas}}})$ and $\mathcal{D}(\mathbb{G}^{an,\text{dR}}_{a,\mathbb{C}_{\text{gas}}})$ in terms of certain modules over the Weil algebra of differential operators, so called *D-modules.*

As a first step, we must then understand $\mathcal{D}(*/\widehat{\mathbb{G}_{a,\mathbb{C}_{\text{gas}}}})$ and $\mathcal{D}(*/\mathbb{G}^\dagger_{a,\mathbb{C}_{\text{gas}}})$. From usual algebraic geometry, we know to expect $QCoh(*/G) = \text{Rep}_G$ for any algebraic group, this is true in large generality:

Theorem 25. *Let* $\mathcal{D} : Corr(\mathcal{C}, E)^\otimes \to Cat$ *be a 6-functor formalism and* $f : G \to 1_\mathcal{C}$ *a group object in* $\mathcal{C}$ *such that* f *is cohomologically proper*[19]*, then* $f_* 1_G$ *has the structure of a coalgebra and there is an equivalence of* ∞*-categories*

$$\mathcal{D}(1_\mathcal{C}/G) \simeq coMod_{f_* 1_G}\mathcal{D}(1_\mathcal{C}).$$

Dually, if f *is cohomologically smooth with dualising sheaf, set* $f_\natural - = f_!(L) \otimes -$*. Then* $f_\natural 1_G$ *has the structure of an algebra and there exists an equivalence of* ∞*-categories*

$$\mathcal{D}(1_\mathcal{C}/G) \simeq \text{Mod}_{f_\natural 1_G}\mathcal{D}(1_\mathcal{C})$$

[18]This is the same as the (automatically open) !-image of the cohomologically smooth map $\Delta : X \to X \times X$

[19]In the proof we will only require that 1_G is f-proper and the proper dual is invertible.

Proof. Using that $1_{\mathcal{C}} \to 1_{\mathcal{C}}/G$ is split and thus a !-cover, we see that the morphism $h : 1_{\mathcal{C}} \to 1_{\mathcal{C}}/G$ is also cohomologically proper. One checks, by the Barr-Beck-Lurie theorem, that the adjunction $h^* \dashv h_*$ is comonadic. Using, that h is cohomologically proper, we can use the projection formula to see that

$$h^*h_*(K) \simeq f_*f^*(K) \simeq f_*(f^*(K) \otimes 1_G) \simeq K \otimes f_*(1_G).$$

A slightly more sophisticated computation shows, that this induces an isomorphism of comonads between h^*h_* and the comonad structure on $- \otimes f_*1_G$ given by an induced coalgebra structure on f_*1_G. □

Next, we will understand what such a coalgebra structure amounts to using Cartier duality.

Lemma 26. *There is an equivalence of* ∞*-categories*

$$\mathcal{D}(*/\widehat{\mathbb{G}_{a,\mathbb{C}_{\text{gas}}}}) \simeq \mathcal{D}(\mathbb{C}_{\text{gas}}[U])$$

given by sending a representation $M \to M \otimes_{\mathbb{C}} \mathbb{C}[[U]]$ *to* $\mathbb{C}_{\text{gas}}[U]$*-module* M *on which* U *acts by* $M \to M \otimes_{\mathbb{C}} \mathbb{C}[[U]] \overset{id\otimes\pi_{U^1}}{\to} M$*; conversely send a* $\mathbb{C}_{\text{gas}}[U]$*-module* (M, f) *to the representation* $c : M \to M \otimes_{\mathbb{C}} \mathbb{C}[[U]]$ *given by* $m \mapsto \exp(Uf)(m) = \sum \frac{U^i}{i!} f^i(m)$*.*

This equivalence intertwines convolution on $\mathcal{D}(*/\widehat{\mathbb{G}_{a,\mathbb{C}_{\text{gas}}}})$ *with the tensor product on* $\mathcal{D}(\mathbb{C}_{\text{gas}}[U])$*. The tensor product of* $\mathcal{D}(*/\widehat{\mathbb{G}_{a,\mathbb{C}_{\text{gas}}}})$ *is intertwined with the operation* $\otimes'$ *on* $\mathcal{D}(\mathbb{C}_{\text{gas}}[U])$ *sending two* $\mathbb{C}_{\text{gas}}[U]$*-modules* M, N *on which* U *acts via* f *and* g *respectively to the* $\mathbb{C}_{\text{gas}}[U]$*-module* $M \otimes_{\mathbb{C}_{\text{gas}}} N$ *on which* U *acts via* $m \otimes n \mapsto f(m) \otimes n + m \otimes g(n)$*.* [20]

Proof. It is easy to check that the described functors are well-defined and that the second composed with the first yields the identity functor. It remains to check that any representation $c : M \to M \otimes_{\mathbb{C}_{\text{gas}}} \mathbb{C}_{\text{gas}}[[U]]$ is determined by its projection to first coefficient of the power series. Projecting on the n-th coefficient yields a map $f_n : M \to M$. The fact that c is a coaction map translates on the corresponding power series in $F := \sum f_n U^n \in End(M)[[U]]$ to the conditions $F(x + y) = F(x)F(y)$ and $F(0) = 0$. This implies $n!f_n \cdot m!f_m = (n + m)!f_{n+m}$. In characteristics 0, this shows that $F = \exp(f_1 U)$. □

Remark 27. *As* $\widehat{\mathbb{G}_{a,\mathbb{C}_{\text{gas}}}} \subset \mathbb{G}_{a,\mathbb{C}_{\text{gas}}}$ *is an open immersion and* $\mathbb{G}_{a,\mathbb{C}_{\text{gas}}}$ *is cohomologically smooth with dualising sheaf* $\mathcal{O}_{\mathbb{G}_{a,\mathbb{C}_{\text{gas}}}}[2]$[21]*,* $\widehat{\mathbb{G}_{a,\mathbb{C}_{\text{gas}}}}$ *is also cohomologically smooth with dualising sheaf* $\mathcal{O}_{\widehat{\mathbb{G}_{a,\mathbb{C}_{\text{gas}}}}}[2]$*. So we could have also proved this statement purely abstractly. But this presentation gives us a clear indication for what* $\mathcal{D}(*/\mathbb{G}^{\dagger}_{a,\mathbb{C}_{\text{gas}}})$ *should look like in terms of* $\mathbb{C}_{\text{gas}}[U]$*-modules: Intuitively, we have to impose a growth condition on* $f : M \to M$ *such that* $\exp(fU) : M \to M \otimes \mathbb{C}[[U]]$ *factors over the ring of germs of holomorphic functions, i.e. for each* $m \in M$

$$\sum_{n\in\mathbb{N}} \frac{f^n(m)}{n!} U^n$$

converges on a small open neighborhood.

Corollary 28. *There is an equivalence of categories*

$$\mathcal{D}(\mathbb{G}^{\text{dR}}_{a,\mathbb{C}_{\text{gas}}}) \simeq \mathcal{D}(\mathbb{C}_{\text{gas}}[\partial, T]_{ass}/(\partial T - T\partial - 1)).$$

[20] Using the fact that $\widehat{\mathbb{G}_{a,\mathbb{C}_{\text{gas}}}}$ is also cohomologically smooth, one could in fact have used another purely formal argument for this statement: Applying the theorem in the opposite 3-functor formalism, one obtains for a cohomologically smooth group object G that $\mathcal{D}(1_{\mathcal{C}}/G) \simeq \text{Mod}_{f_\natural 1_G}\mathcal{D}(1_{\mathcal{C}})$.

[21] If we further embedded into $\mathbb{P}^1_{\mathbb{C}_{\text{gas}}}$, one could prove this using ordinary Grothendieck duality.

Proof. $\pi : \mathbb{G}^{\mathrm{dR}}_{a,\mathbb{C}_{\mathrm{gas}}} \to */\widehat{\mathbb{G}_{a,\mathbb{C}_{\mathrm{gas}}}}$ is affine, whence

$$\mathcal{D}(\mathbb{G}^{\mathrm{dR}}_{a,\mathbb{C}_{\mathrm{gas}}}) \simeq Mod_{\pi_* 1_{\mathbb{G}^{\mathrm{dR}}_{a,\mathbb{C}_{\mathrm{gas}}}}} \mathcal{D}(*/\widehat{\mathbb{G}_{a,\mathbb{C}_{\mathrm{gas}}}}).$$

$\pi_* 1_{\mathbb{G}^{\mathrm{dR}}_{a,\mathbb{C}_{\mathrm{gas}}}}$ corresponds to the $\mathbb{C}_{\mathrm{gas}}[[U]]$-comodule $\mathbb{C}_{\mathrm{gas}}[T] \to \mathbb{C}_{\mathrm{gas}}[T] \otimes_{\mathbb{C}_{\mathrm{gas}}} \mathbb{C}_{\mathrm{gas}}[[U]]$ sending $T^n \mapsto (T \otimes 1 + 1 \otimes U)^n = nT^{n-1} \otimes U + \dots$. Under the previous equivalence, this corresponds to the $\mathbb{C}_{\mathrm{gas}}[U]$-module structure on the $\mathbb{C}_{\mathrm{gas}}$-module $\mathbb{C}_{\mathrm{gas}}[T]$, where U acts by differentiation i.e. $T^n \mapsto nT^{n-1}$. This is an algebra for $\otimes'$ using the multiplication $f \otimes g \mapsto fg$ [22]. Now a module for this algebra is a $\mathbb{C}_{\mathrm{gas}}[U]$-module M, say U acts via f, together with an action map $\mathbb{C}_{\mathrm{gas}}[T] \otimes' M \to M$, say T acts via g. Then by definition of $\otimes'$, this is equivalent endomorphisms f and g such that $fg = 1 + gf$.[23] □

Theorem 29. *In general, let X be a complex analytic space, then*

$$\mathcal{D}(X_{\mathrm{dR}}) \simeq \mathcal{D}(\mathrm{Mod}_{D_X})$$

for D_X the sheaf of differential operators.

2.5 Comparisons between deRahm stacks

Our next goal is to more concretely understand $\mathcal{D}(X^{an}_{\mathrm{dR}})$ and especially $\mathcal{D}(\mathbb{G}^{an}_{a,\mathbb{C}_{\mathrm{gas}},\mathrm{dR}})$.

Proposition 30. *Let $g : */\widehat{\mathbb{G}_{a,\mathbb{C}_{\mathrm{gas}}}} \to */\mathbb{G}^{\dagger}_{a,\mathbb{C}_{\mathrm{gas}}}$ be the canonical map. The sheaf $\mathcal{O}_{*/\widehat{\mathbb{G}_{a,\mathbb{C}_{\mathrm{gas}}}}}$ is g-proper with g-proper dual $\mathcal{O}_{*/\widehat{\mathbb{G}_{a,\mathbb{C}_{\mathrm{gas}}}}}[-2]$. Further,*

$$g^* : \mathcal{D}(*/\mathbb{G}^{\dagger}_{a,\mathbb{C}_{\mathrm{gas}}}) \to \mathcal{D}(*/\widehat{\mathbb{G}_{a,\mathbb{C}_{\mathrm{gas}}}}) \simeq \mathcal{D}(\mathbb{C}_{\mathrm{gas}}[U])$$

is fully faithful with essential image given by those modules killed after tensoring with the idempotent $\mathbb{C}_{\mathrm{gas}}[U]$-algebra of those power series

$$\sum_{n \in \mathbb{Z}} a_n U^n \in \mathbb{C}((U^{-1}))$$

for which there exists some $r > 0$ such that $|a_n| \frac{r^n}{n!} \to 0$.

Proof. Consider the maps

$$* \xrightarrow{h} */\widehat{\mathbb{G}_{a,\mathbb{C}_{\mathrm{gas}}}} \xrightarrow{g} */\mathbb{G}^{\dagger}_{a,\mathbb{C}_{\mathrm{gas}}}, \qquad q : * \to */\mathbb{G}^{\dagger}_{a,\mathbb{C}_{\mathrm{gas}}}.$$

By !-descent, as $\mathbb{G}^{\dagger}_{a,\mathbb{C}_{\mathrm{gas}}} \to \mathrm{AnSpec}\mathbb{C}_{\mathrm{gas}}$ is proper, we know that $\mathcal{O}$ is q-proper with q-proper dual $\mathcal{O}_*$. Further, by !-descent, h is cohomologically smooth and surjective with dualising sheaf $\mathcal{O}[2]$. By the lemma in the beginning, it follows that $h_! \mathcal{O}$ is g-proper.

There is an exact triangle

$$h_! \mathcal{O}[1] \xrightarrow{U} h_! \mathcal{O}[1] \longrightarrow \mathcal{O}$$

this is true as $h_! \mathcal{O}[1]$ corresponds to the $\mathbb{C}_{\mathrm{gas}}[U]$-module $\mathbb{C}[T^{\pm}]/\mathbb{C}[T]$ with U acting by differentiation (which is injective), $\mathcal{O}$ corresponds $\mathbb{C}_{\mathrm{gas}}$ and there is a short exact sequence

[22] This follows from the Leibniz rule.

[23] One could also have proven this entirely differently by using a similar comonadicity statement to show $\mathcal{D}(\mathbb{G}^{\mathrm{dR}}_{a,\mathbb{C}_{\mathrm{gas}}}) \simeq coMod_{\sigma_* 1}\mathcal{D}(\mathbb{G}_{a,\mathbb{C}_{\mathrm{gas}}})$, where $\sigma :_{\mathbb{G}_{a,\mathbb{C}_{\mathrm{gas}}} \times \widehat{\mathbb{G}_{a,\mathbb{C}_{\mathrm{gas}}}}} \to \mathbb{G}_{a,\mathbb{C}_{\mathrm{gas}}}$ is the action map and then arguing as before.

$$0 \longrightarrow \mathbb{C}[T^{\pm}]/\mathbb{C}[T] \xrightarrow{U} \mathbb{C}[T^{\pm}]/\mathbb{C}[T] \longrightarrow \mathbb{C} \longrightarrow 0.$$

Hence $\mathcal{O}$ is also g-proper. In particular, this implies that g_* satisfies the projection formula.

Now we prove fully faithfulness of g, we need to check that $id \to g_* g^*$ is an isomorphism. But as $g_* g^*$ satisfies the projection formula, it suffices to check that $g_* \mathcal{O} \simeq \mathcal{O}$.

Applying $g_!$ to the previous triangle, we get a triangle

$$(g \circ h)_! \mathcal{O}[1] \xrightarrow{U} (g \circ h)_! \mathcal{O}[1] \longrightarrow g_! \mathcal{O}$$

but $q = g \circ h$ is proper and thus $q_! \mathcal{O} = q_* \mathcal{O}$ is the regular representation of $\mathbb{G}^{\dagger}_{a,\mathbb{C}_{\text{gas}}}$. U acts by differentiation on it, which is surjective with the kernel given by constant functions, so we get a triangle

$$\mathcal{O}[1] \longrightarrow (g \circ h)_! \mathcal{O}[1] \xrightarrow{U} (g \circ h)_! \mathcal{O}[1]$$

Rotating the first triangle shows that $g_! \mathcal{O}[-1] \simeq \mathcal{O}[1]$. Thus, $g_! \mathcal{O}[2] \simeq \mathcal{O}$. In the same way one calculates the g-proper dual[24] using the formulas from the beginning to be $\mathcal{O}[2]$. Hence $g_* \mathcal{O} \simeq g_! \mathcal{O}[2] \simeq \mathcal{O}$, which proves that g is fully faithful.

The monoidal unit being g-proper with invertible g-proper dual implies that the adjunction $g^* g_*$ is comonadic. A calculation using the projection formula shows that we have an isomorphism of comonads:

$$g^* g_*(-) \simeq g^* g_*(\mathbb{C}_{\text{gas}}[U]) \star -$$

where $\star$ is the convolution product. This implies

$$\mathcal{D}(*/\mathbb{G}^{\dagger}_{a,\mathbb{C}_{\text{gas}}}) \simeq co\text{Mod}_{g^* g_* \mathbb{C}_{\text{gas}}[U]} \mathcal{D}(*/\widehat{\mathbb{G}_{a,\mathbb{C}_{\text{gas}}}}).$$

As g^* is fully-faithful, the coalgebra

$$g^* g_*(\mathbb{C}_{\text{gas}}[U])$$

is automatically idempotent. Having the structure of a comodule over an idempotent coalgebra is equivalent to killing $\text{fib}(g^* g_*(\mathbb{C}_{\text{gas}}[U]) \to \mathbb{C}_{\text{gas}}[U])$. We know $\mathbb{C}_{\text{gas}}[U] = h_! \mathcal{O}[1]$ whence $g^* g_* h_! \mathcal{O}[1] \simeq g^* q_* \mathcal{O}[-1]$ corresponds to the regular representation of $\mathbb{G}^{\dagger}_{a,\mathbb{C}_{\text{gas}}}$ viewed as a mere $\widehat{\mathbb{G}_{a,\mathbb{C}_{\text{gas}}}}$-representation and shifted into cohomological degree 1.

The cone now becomes an extension between the module of germs of holomorphic functions $\{f \in \mathbb{C}[[T]] | f \text{ converges on a small disk}\}$, where U acts by differentiation and $\mathbb{C}[T^{\pm 1}]/\mathbb{C}[T]$, where U also acts by differentiation. It is the module

$$\{f \in \mathbb{C}((T)) | f \text{ converges on a small punctured disk}\}$$

U acts invertibly on this and we can write $T^n = n! U^{-n}(1)$. A power series $\sum a_n T^n$ converges on a small punctured disk if and only if there exists $r > 0$ such that $|a_n| r^n \to 0$. Writing the module in the natural basis U^n, this gives

$$\left\{ \sum_{n \in \mathbb{Z}} a_n U^n \in \mathbb{C}((U^{-1})) \; \exists r > 0 \text{ such that } |a_n| \frac{r^n}{n!} \to 0 \right\}.$$

□

[24]The proper dual of $h_! \mathcal{O}[1]$ is itself, so we may apply the proper dual functor and compute.

As mentioned last time, the map $j : \mathbb{G}^{an}_{a,\mathbb{C}_{\text{gas}}} \hookrightarrow \mathbb{G}_{a,\mathbb{C}_{\text{gas}}}$ is an open immersion i.e. we have a fully faithful embedding

$$j_! : \mathcal{D}(\mathbb{G}^{an}_{a,\mathbb{C}_{\text{gas}}}) \to \mathcal{D}(\mathbb{C}_{\text{gas}}[T])$$

whose image is given by those modules killed by the idempotent algebra of those power series

$$\sum_{n\in\mathbb{Z}} b_n T^n \in \mathbb{C}((T^{-1}))$$

that converge on a small punctured disk around ∞. Together these results assemble to the fact that the pullback functor

$$\mathcal{D}(\mathbb{G}^{an}_{a,\mathbb{C}_{\text{gas}},\text{dR}}) \to \mathcal{D}(\mathbb{G}^{\text{dR}}_{a,\mathbb{C}_{\text{gas}}}) \simeq \mathcal{D}(\mathbb{C}_{\text{gas}}[\partial, T]_{ass}/(\partial T - T\partial - 1))$$

is fully faithful.

Ignoring the fact the Weil algebra is non-commutative, one can imagine this as the category of quasi-coherent sheaves on an open subset of $\mathbb{A}^2_{\mathbb{C}_{\text{gas}}}$ where in both directions $T \to \infty$ and $\partial \to \infty$, we get the growth condition that are explicitly described idempotent algebras of functions near ∞ must be killed.

Remark 31. *For a general complex manifold, this theorem generalises. Denote by $g_X : X_{\text{dR}} \to X^{an}_{\text{dR}}$ the canonical map. Them the sheaf $\mathcal{O}_{X_{\text{dR}}}$ is g_X-proper with g_X-proper dual $\mathcal{O}_{X_{\text{dR}}}[-2d_X]$, where d_X is the complex dimension of X and the functor*

$$g_X^* : \mathcal{D}(X^{an}_{\text{dR}}) \to \mathcal{D}(X_{\text{dR}}) \simeq \mathcal{D}(\text{Mod}_{D_X})$$

is fully faithful.

3 Analytic Riemann-Hilbert

Theorem 32. *(Analytic Riemann-Hilbert) The morphism $X \to X(\mathbb{C})_{\text{Betti}}$ factors over X^{an}_{dR} and the induced map $X^{an}_{\text{dR}} \to X(\mathbb{C})_{\text{Betti}}$ is an isomorphism.*

Proof. This is essentially per definition. We need to compare the equivalence relation $X \times_{X(\mathbb{C})_{\text{Betti}}} X \to X \times X$ with $\Delta(X)^\dagger$. This follows from the general fact, that for $Z \subset X$,

$$(Z \times X)^\dagger = X \times_{X_{\text{Betti}}} Z_{\text{Betti}}$$

which follows from definition as

$$Z_{\text{Betti}} = \varprojlim_{Z\subset U} U_{\text{Betti}}.$$

□

3.1 Relation to usual Riemann-Hilbert

The classical Riemann-Hilbert correspondence can now be explained by exhibiting a partial inverse functor to g^*:

Theorem 33. *On the subcategory of* regular holonomic *D-modules*

$$\mathcal{D}^{\text{rh}}(X_{\text{dR}}) \subset \mathcal{D}(\text{Mod}_{D_X})$$

the funcotr g_* is fully faithful and the induced functor

$$\mathcal{D}^{\text{rh}}(X_{\text{dR}}) \hookrightarrow \mathcal{D}(X^{an}_{\text{dR}}) \simeq \mathcal{D}(X(\mathbb{C}))$$

has image given by complexes with Zariski-constructible cohomology. Moreover, the functor $g_*[d_X]$[25] is t-exact for the usual t-structure on the left and the perverse t-structure on the right.

[25]This is the middle degree between g_* and $g_! = g_*[2d_X]$.