Notes - Arakelov Geometry and Condensed Mathematics

Contents

Foreword.

These are notes for the workshop “Arakelov Geometry and Condensed Mathematics” which took place in Strasbourg, May 19-23 2025. See:

`https://irma.math.unistra.fr/~lfu/Activities/Conference-Arakelov-and-Condensed.html`

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1 Introduction to Arakelov geometry, Part 1 - by Jean-Benoît Bost

There are 4 slogans:

(1) Arakelov geomometry is a higher dimensional version of the classical analogy between number fields and function fields

(2) It is a hybrid theory that combines algebraic geometry (Grothendieck style) over Spec$\mathbb{Z}$, and complex Hermitian geometry (of smooth complex projective varieties). Its main object of studies are Hermitian vector bundles and related "mixtures" of algebraic and Hermitian differential geometric objects (e.g. Arakelov divisors)

(3) Arakelov geometry attaches to these objects *real valued* invariants that play the role of the integer valued invariants of classical algebraic geometry (plurigenera, intersection numbers).

(4) Arakelov geometry may be used to establish theorems of Diophantine geometry admitting "elementary" formulations. For example on rational points of algebraic varieties over number fields (Falting, Vojta), or to obtain transcendance results.

Here is an example of real valued invariant which is important in Arakelov geometry.

Definition 1.1 (Height)**.** Let $P \in \mathbb{P}^N(\mathbb{Q})$, the height of P is the arithmetic complexity of P. Denote

$$P = [x_0 : x_1 : \ldots : x_N]$$

with $x_i \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_N) = 1$. Then

$$\begin{aligned} h(P) &= \log(\max_{0 \leqslant i \leqslant N} |x_i|) \\ &= \log(\sum x_i^2)^{1/2} + O(1) \\ H(P) &= \exp(h(P)). \end{aligned}$$

Remark 1.2. When one want to store an integer, a rational number, or a point of the projective space in a computer, one would like to know how much space to allocate. That is, how much bits would it take to store the data in the memory of the machine ? The notion of height h defined above give a good order of magnitude for such quantities.

1.1 Analogy between number fields and function fields

Let us first set up notations for both sides of the analogy

1) Given a base field k, a function field is a finite extension K of $k(T)$, we will denote d its degree.

2) A number field is a finite extension K of $\mathbb{Q}$, we will denote d its degree.

Assume that k is algebraically closed in K, then $K = k(C)$ where C is a smooth projective curve, geometrically connected over k. Moreover, as K is an extension of $k(T)$, we get a map $C \to \mathbb{P}^1_k$. We will define $\mathring{C}$ and Δ as fiber products

$$\begin{tikzcd}
\mathring{C} \arrow[r] \arrow[d] & C \arrow[d] & \Delta \arrow[l] \arrow[d] \\
\mathbb{A}^1_k \arrow[r] & \mathbb{P}^1_k & \operatorname{Spec}(k) \arrow[l]
\end{tikzcd}$$

The construction of $\mathring{C}$ is analogous to the construction of $\mathcal{O}_K$ as the integral closure of $\mathbb{Z}$ in K in the number field case.

Remark 1.3. Gauss discovered and studied extensively the class group of a number field. Over quadratic fields, the class group can be phrased in terms of quadratic forms. So for many years, people were mislead to find a good generalization of class group of higher degree fields.

Remark 1.4. For a long time, the theory of complex curves developped by Riemann was analytic in nature. In particular the construction of curve was done analytical locally.

We will only consider $k = \mathbb{C}$ from now on.

Remark 1.5. Often in this analogy, the function field side is seen as easy, whereas the number field side is considered complicated. But somehow, the results where sometimes first discovered in the number field case, to then be developped in the theory of function fields.

Let K be a number field, the next step of the analogy is to consider $\operatorname{Spec}(\mathcal{O}_K)$ to be an affine curve $\mathring{C}$.

Remark 1.6. Assume $k = \mathbb{C}$. Given $x \in C(\mathbb{C})$, we have $v_x : K \to \mathbb{Z} \cup \{+\infty\}$ mapping a rational function to its order of vanishing at x. It satisfies $v_x(\mathbb{C}^\star) = 0$. This gives a correspondance

$$\{\text{points of C}\} \longleftrightarrow \{\text{valuations on } K\}$$

Consider $\mathbb{C}(T)$, one has the valuation $v_\infty\left(\frac{P}{Q}\right) = -\deg(P) + \deg(Q)$, corresponding to the point "at infinity" in the projective line.

This allows to spell out more precisely the analogy

$$\begin{array}{lcr}
\mathbb{C}(T) & & \mathbb{Q} \\
v_x : \ x \in \mathbb{C} & \longleftrightarrow & v_p : \ p \text{ prime number} \\
v_\infty & & |\cdot|
\end{array}$$

Or more generally for a number field K and the function field of a smooth projective curve $C = \mathring{C} \cup \Delta$,

$$\begin{array}{lcc} \mathbb{C}(C) & & K \\ v_x : x \in \mathring{C} & \longleftrightarrow & v_{\mathfrak{p}} : \mathfrak{p} \text{ prime ideal} \\ v_\infty : \infty \in \Delta & & \text{Archimedean valuations on } K \end{array}$$

Let K be a number field, and $\sigma : K \hookrightarrow \mathbb{C}$ be a complex embedding. One has an associated valuation $|x|_\sigma = |\sigma(x)|$. This construction gives rise to a bijection between the set of archimedean valuations on K and

$$\{\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathcal{O}_K)\}/\text{complex conjugation}$$

Weil, Artin and Hasse discovered a Rosetta stone, allowing one to translate and discover meaningful results in the context of the analogy:

1) In the function field case with $k = \mathbb{C}$, the theory has been well developped by Riemann. One has the notion of genus of a curve, which is topological in nature, and appears in Riemann-Roch theorem

2) In the function field case with $k = \mathbb{F}_q$, one still has a Riemann-Roch theorem, hence a notion of genus for curves. One also has a notion of ζ functions and L-functions.

3) In the case of a number field K, one can also defined ζ functions and L functions. They are in fact defined for any scheme of finite type over $\mathbb{Z}$.

   The quantity analogous to $2g - 2$ in this case is $\log|\Delta_K|$.

   The analytic continuation of ζ function was first obtained for number fields, which is the "difficult case", and it was then adapted to curves.

Remark 1.7. Other more recent research domain are also in the spirit of the analogy described here. This is notably the case for Iwasawa theory, which is inspired by the work of Weil. It would be a worth long time goal to spill out more precisely the dictionary between geometry and number theory in this case.

1.2 Hybrid between $\mathrm{Spec}(\mathbb{Z})$ and Hermitian geometry over $\mathbb{C}$

We first set up the notations. Let K be a number field and X be a closed reduced and geometrically integral subvariety of $\mathbb{P}^N_K$, defined by an homogenous ideal $I(X) \subset K[X_0, \ldots X_N]$. We will consider

$$\mathcal{X} := \text{Zariski closure of } X \text{ in } \mathbb{P}^N_{\mathcal{O}_K}.$$

Then $\mathcal{X}$ is defined by the ideal

$$I(\mathcal{X}) := I(X) \cap \mathcal{O}_K[X_0, \ldots, X_N].$$

Given a nonzero prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$, we denote $\mathbb{F}_\mathfrak{p}$. Then we consider

$$\mathcal{X}_{\mathbb{F}_\mathfrak{p}} = \mathcal{X} \otimes_{\mathcal{O}_K} \mathbb{F}_\mathfrak{p}.$$

The corresponding homogenous ideal is

$$I(\mathcal{X}_{\mathbb{F}_\mathfrak{p}}) = I(\mathcal{X})/\mathfrak{p}I(\mathcal{X}) \subset \mathbb{F}_\mathfrak{p}[X_0, \dots, X_N].$$

We can do the same with $\mathfrak{p}$ replaced by $\sigma : K \to \mathbb{C}$, in which case

$$I(X_\sigma) = \sigma(I(X)) \subset \mathbb{C}[X_0, \dots, X_N].$$

Denote by A the set of complex embeddings of K:

$$A = \mathrm{Hom}(K, \mathbb{C}) = (\mathrm{Spec}\, K)(\mathbb{C}).$$

We have the equalities $|A| = d = r_1 + 2r_2$, where d is the degree of K, r_1 is the number of real embeddings, and r_2 is the number of complex embeddings up to complex conjugation. An improvement of the previous equality is the following isomorphism

$$\begin{aligned} K \otimes_\mathbb{Q} \mathbb{C} &\to \mathbb{C}^A \\ x \otimes \lambda &\mapsto (\sigma(x)\lambda)_{\sigma \in A} \end{aligned}$$

which yields $K \otimes_\mathbb{Q} \mathbb{R} \cong (\mathbb{C}^A)^{\text{c.c.}}$, where c.c. denotes complex conjugation.

Moreover, we have the identifications

$$X_\mathbb{C} = X \otimes_\mathbb{Q} \mathbb{C} = X \otimes_K (K \otimes_\mathbb{Q} \mathbb{C}).$$

As $K \otimes_\mathbb{Q} \mathbb{C} = \mathbb{C}^A$, $X_\mathbb{C}$ decomposes as

$$X_\mathbb{C} \cong \biguplus_{\sigma \in A} X_\sigma.$$

Let $\mathcal{X}$ be a separeted scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$.

Definition 1.8. An Hermitian vector bundles over $\mathcal{X}$ is a pair $\overline{E} = (E, ||.||)$ where

- E is a vector bundle over $\mathcal{X}$
- $||.||$ is an Hermitian norm on the $\mathcal{C}^\infty$ vector bundle E^{an} on $\mathcal{X}(\mathbb{C})$, compatible with complex conjugation.

Basic operations on Hermitian vector bundles:

1) Given $f : \mathcal{X}' \to \mathcal{X}$, one has an Hermitian vector bundle $f^\star \overline{E}$ over $\mathcal{X}'$,
2) One has the usual tensor operations: given $\overline{E}$ and $\overline{F}$, we can build $\overline{E} \oplus \overline{F}$, $\bigwedge^i \overline{E}$, $\mathrm{Sym}^i \overline{E}$, $\overline{E} \otimes \overline{F}$.

1.3 Arakelov divisors

Assume that $\mathcal{X}$ is integral, with generic point η. Let $\overline{L} = (L, ||\cdot||)$ be an Hermitian line bundle on $\mathcal{X}$. Given $s \in \Gamma(X_\eta, L_\eta) \smallsetminus \{0\}$ a non-zero meromorphic section, one gets a pair

$$\left(\mathrm{div} s, \log(||s||^{-1})\right)$$

where $D = \mathrm{div} s$ is a Cartier divisor and $\log(||s||^{-1})$ is a $\mathcal{C}^\infty$ function on $\mathcal{X}(\mathbb{C}) \smallsetminus D(\mathbb{C})$ which has logarithmic singularities along $D_\mathbb{C}$. We will call such a function a Green function for $D_\mathbb{C}$ in $\mathcal{X}(\mathbb{C})$.

Definition 1.9 (Arakelov divisors)**.**

1) A pair (D, g) where g is a Green function for $D_\mathbb{C}$ is called an Arakelov divisor.
2) Given $f \in K(\mathcal{X})^\star$, we have $\widehat{\mathrm{div}} f = (\mathrm{div} f, \log|f_\mathbb{C}|)$. We call such an Arakelov divisor a principal Arakelov divisor.

We will consider

$$\overline{\mathcal{O}}_\mathcal{X} := (\mathcal{O}_\mathcal{X}, ||\cdot|| = 1),$$

and for any Arakelov divisor (D, g)

$$\overline{\mathcal{O}}_\mathcal{X}(D, g) := (\mathcal{O}_\mathcal{X}(D), ||\cdot|| = e^{-g})$$

This induces bijections

$$\{(\overline{L}, s)\}/\mathrm{isom} \cong \{\text{Arakelov divisors } (D, g) \text{ in } \mathcal{X}\}$$

$$\{\overline{L} \text{ Hermitian line bundle}\}/\mathrm{isom} \cong \{\text{Arakelov divisors } (D, g) \text{ in } \mathcal{X}\}/\text{principal divisors.}$$

Definition 1.10. We will denote by $\widehat{\mathrm{Pic}}(\mathcal{X})$ the group

$$\{\overline{L} \text{ Hermitian line bundle}\}/\mathrm{isom}.$$

Example 1.11. For $\mathcal{X} = \mathrm{Spec}\mathbb{Z}$, we have $\overline{E} = (E, ||.||)$ where $E \cong \mathbb{Z}^N$ and $||\cdot||$ is a Euclidean norm on $E_\mathbb{R} \cong \mathbb{R}^N$. Hence the analogy gives more concretely in this setting:

$$\text{Euclidean lattices} \longleftrightarrow \text{vector bundles over projective curves.}$$

Let K be a function field, and $\mathcal{X} = \mathrm{Spec}\mathcal{O}_K$. In this case

$$\overline{E} = (E, (||\cdot||_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}}),$$

where $||\cdot||_\sigma$ is a norm on E_σ. One gets $\mathrm{Pic}(\mathcal{X}) = \mathrm{Cl}(K)$ and $\widehat{\mathrm{Pic}}(\mathcal{X})$ has a surjective map to $\mathrm{Cl}(K)$ whose kernel is

$$\mathbb{R}^{r_1+r_2}/\log(|\mathcal{O}_K^\star|) \cong \mathbb{R} \times \mathbb{T}^{r_1+r_2-1}.$$

2 Analytic rings I - by Arthur-César Le Bras

2.1 What is analytic geometry?

The word is overused, here we mean it in the sense of Clausen-Scholze.

The first kind of geometry one encounters is differential geometry, but then one encounters complex analytic geometry. Then we learn algebraic geometry, but also the p-adic analog of complex analytic geometry.

All these theories are similar but slightly different, making it unclear that they go in a common framework. The goal of analytic geometry here is to give such a framework.

Goal: We want to do analytic geometry like we do algebraic geometry.

In algebraic geometry, the functions are polynomials, hence we have many tools to study them, such as homological algebra, which are not readily available for rings of continuous or analytic functions.

Recall the way one classically builds the theory of schemes. It proceeds in two steps.

Step 1 (Affine schemes): We want to associate to any ring R a topological space $\mathrm{Spec}(R)$ endowed with a sheaf of ring.

Step 2 (Glueing): A scheme is a locally ringed space which is locally of the form $\mathrm{Spec}(R)$.

Alternative point of view:

$$\mathrm{AffSch} = \mathrm{Ring}^{\mathrm{op}} + \text{Zariski topology}$$

A scheme is then a sheaf for the Zariski topology which is locally affine. This perspective is useful for generalizations: we can consider sheaves of groupoids, spaces and so on.

One can also replace Zariski topology by étale topology, fppf topology ...

One of the requests is to choose a Grothendieck topology τ such that the abelian categories $\mathrm{Mod}(R)$, or at least the derived categories $D(R) = D(\mathrm{Mod}(R))$, glue. This glueing condition is also called *descent*.

This leads to a notion of $D(X)$ for X a scheme or the appropriate generalization. This is by definition the derived category of quasi-coherent sheaves on X, and it satisfies $D(\mathrm{Spec}(R)) = D(R)$.

Plan for the lectures:

1. Analogue of Step 1 → Analytic affine stack
2. Analogue of Step 1, continued
3. Analogue of Step 2, choice of topology → Analytic stacks
4. The last 3 talks will be about examples of analytic stacks

Reference: Main course of Clausen-Scholze "Analytic Stacks" (2023-24) on Youtube [3].

Today: Condensed mathematics on Scholze's webpage (careful, there are some differences) Topoi and condensed sets (Anschütz).

2.2 Light condensed sets

Reference: [3, Lectures 2 and 3]

Naive attempt: use topological rings, and consider topological modules over your topological ring. But all techniques of homological algebra, derived categories, derived functors, are difficult to use in presence of a topology...

Basic example: take $\mathbb{Z}$ as a discrete ring. The category of topological abelian groups is not abelian ! The standard example is that $\mathrm{id} : \mathbb{R}_{\mathrm{disc}} \to \mathbb{R}_{\mathrm{nat}}$ has trivial kernel and no cokernel but it is not an isomorphism.

Idea: we need to change the notion of topological space. This is why Clausen-Scholze defined (light) condensed sets.

Definition 2.1. A *profinite set* is a compact Hausdorff space which is totally disconnected. They are also equivalently described as cofiltered limits of finite sets.

$\mathrm{Pro}(\mathrm{Fin}) \cong \mathrm{Prof} \subset \mathrm{Top}$ is the category of profinite sets. It is stable under limits.

Definition 2.2. A profinite set S is *light* if it is metrizable. Light profinite sets are characterized as the sequential limits of finite sets, or as profinite sets such that $\mathrm{Cont}(S, \mathbb{Z})$ is countable. We denote by $\mathrm{Prof}^{\mathrm{light}}$ the category of light profinite sets.

Example 2.3.
- finite sets
- $\mathbb{N} \cup \{\infty\} = \varprojlim \{0, 1 \dots, n, \infty\}$ where the transition maps
$$\{0, 1, \dots, n, n+1, \infty\} \to \{0, 1, \dots, n, \infty\}$$
are identities on the initial segment and sends $n+1$ to ∞.
- the Cantor set $C = \{0, 1\}^{\mathbb{N}}$

Remark 2.4. Any (metrizable) compact Hausdorff space receives a surjection from a (light) profinite set.

Proof. Let K be a compact Hausdorff, and I be a cofiltered poset of finite open covers $\{U_{i,j} \to K\}_{j \in J_i}$ with $i \in I$ and J_i is finite. For any $i \in I$, $K_i = \biguplus_{j \in J_i} \overline{U_{i,j}}$ is compact Hausdorff. This yields a surjection
$$S = \varprojlim_{i \in I} K_i \to K.$$
And S is totally disconnected (exercise). Moreover one can restrict to only considering the $U_{i,j}$ in a fixed basis of the topology. This gives that S can be taken light if K is metrizable, hence has a countable basis for its topology. $\square$

Definition 2.5. One gets a Grothendieck topology on $\mathrm{Prof}^{\mathrm{light}}$ by declaring that a family

$$\{S_i \to S\}_{i \in I}$$

is a covering if I is finite and $\biguplus_{i \in I} S_i \to S$ is surjective.

Definition 2.6. A light condensed set is a sheaf on $\mathrm{Prof}^{\mathrm{light}}$ endowed with the topology defined above. In other words, it is a functor

$$X : \mathrm{Prof}^{\mathrm{light,op}} \to \mathrm{Set}$$

such that

(i) $X(\varnothing) = \star$

(ii) $X(S_1 \bigcup S_2) = X(S_1) \times X(S_2)$

(iii) for any surjection $S' \to S$, the following map is bijective

$$X(S) \to \{x \in X(S'), p^\star x = q^\star x\},$$

where $p, q : S' \times_S S' \to S'$ are the projections.

The category of light condensed is denoted Cond(Set), it is a topos, and has all limits and colimits. From now on, we will drop the word "light" from the terminology, and everything is assumed to be light.

Let us see the link with usual topological spaces.

$$\begin{aligned} \underline{(-)} : \mathrm{Top} &\to \mathrm{Cond(Set)} \\ T &\mapsto (S \mapsto \mathrm{Cont}(S, T)) \end{aligned}$$

Using that surjective maps of compact Hausdorff spaces are quotient maps, it is an exercise to verify the sheaf property.

Moreover, this functor has a right adjoint.

$$\begin{aligned} \mathrm{Cond(Set)} &\to \mathrm{Top} \\ X &\mapsto X(\star)_{\mathsf{T}} \end{aligned}$$

Explicitely, the underlying set of the topological space associated to X is $X(\star)$ and the topology is the quotient topology

$$\biguplus_{S, x \in X(S)} S \to X(\star).$$

Proposition 2.7. $\underline{(-)}$ *is fully faithful on sequential spaces (quotients of metrizable spaces).*

Intuition: Let $X \in \mathrm{Cond(Set)}$

- One should think of $X(\star)$ as the underlying set of X.
- One should think of $X(\mathbb{N} \cup \{\infty\})$ as the set of convergent sequences in X.

Johnstone already tried to axiomatize these two things, but it has not enough nice categorical properties (if one only uses the profinite set $\mathbb{N} \cup \{\infty\}$, quasi-compact objects will be countable).

2.3 Condensed abelian groups

Reference: [3, Lecture 5]

By considering sheaves of groups, rings, modules, ... one gets the notion of condensed group, ring, module ...

Definition 2.8. Cond(Ab) is the category of sheaves of abelian groups on $\mathrm{Prof}^{\mathrm{light}}$.

It is a Grothendieck abelian category, i.e. it is abelian, has limits and colimits, filtered colimits commute with finite limits.

Example 2.9. In this context we still have the map $\underline{\mathbb{R}_{\mathrm{disc}}} \to \underline{\mathbb{R}_{\mathrm{nat}}}$ induced by identity. It has 0 kernel, but it is not an isomorphism. This can be seen at the level of its cokernel Q:

$$Q : S \mapsto \mathrm{Cont}(S, \mathbb{R})/\mathrm{LocConst}(S, \mathbb{R})$$

Further properties:

- Forgetful functor $\mathrm{Cond(Ab)} \to \mathrm{Cond(Set)}$ has a left adjoint $X \mapsto \mathbb{Z}[X]$. More precisely, $\mathbb{Z}[X]$ is the sheaf associated to the presheaf $S \mapsto \mathbb{Z}[X(S)]$.
- There is a unique closed symmetric monoidal structure on Cond(Ab), denoted $- \otimes -$, commuting with colimits in each variable and such that $X \mapsto \mathbb{Z}[X]$ is symmetric monoidal.

Remark 2.10. One can interpret $\mathbb{Z}[X]$ as the datum of a topology on $\mathbb{Z}[X(\star)]$. Let us make it explicit in one important case:

Given $S \in \mathrm{Prof}^{\mathrm{light}}$, written as $S = \varprojlim_i S_i$ with S_i finite,

$$\mathbb{Z}[S] = \bigcup_n \varprojlim_i \mathbb{Z}[S_i]_{\ell^1 \leqslant n} \subsetneq \varprojlim_i \mathbb{Z}[S_i]$$

Here $\mathbb{Z}[S_i]_{\ell^1 \leqslant n} = \{f : S_i \to \mathbb{Z} \mid \sum_{s \in S_i} |f(s)| \leq n\}$

Proposition 2.11. *$\mathbb{Z}[\mathbb{N} \cup \{\infty\}]$ is internally projective in* Cond(Ab)*, i.e.*

$$\mathcal{H}om_{\mathrm{Cond(Ab)}}(\mathbb{Z}[\mathbb{N} \cup \{\infty\}], -)$$

is exact.

Remark 2.12.

- $\mathbb{N} \cup \{\infty\}$ is not a projective object in Cond(Set). For example
$$(2\mathbb{N} \cup \{\infty\}) \cup (2\mathbb{N} + 1 \cup \{\infty\}) \to \mathbb{N} \cup \{\infty\}$$
is a surjection which has no splitting.
- In the "Old" formalism, one replaces $\mathrm{Prof}^{\mathrm{light}}$ by Prof. $\mathrm{Cond(Ab)}^{\mathrm{old}}$ has a family of compact projective generators, $\mathbb{Z}[S]$ with S extremally disconnected, but they are not internally projective.

- The previous Proposition is not true in the old formalism, and there are no internal projective objects.

More generally, one can defined Cond(Ring) and the category of condensed modules over a condensed ring.

Remark 2.13 (Disclaimer)**.** It seems that we already achieved the goal, that we fixed ourselves. Indeed, we can now replace topological rings and topological modules by their condensed enhancements. But here is the basic problem: let A be a condensed ring, M, N two condensed A-modules

$$(M \otimes_A N)(\star) = M(\star) \otimes_{A(\star)} N(\star).$$

This is the *algebraic* tensor product which appears as the underlying module. But to perform geometric constructions such as product of analytical spaces, or for other purposes in functional analysis, one need a completed tensor product !

2.4 Analytic rings

Reference: [3, Lectures 8 and 9]

Definition 2.14. An *analytic ring* A is a pair $(A^{\triangleright}, D(A))$, where $A^{\triangleright}$ is a condensed ring, $D(A) \subset D(A^{\triangleright})$ is a full triangulated subcategory such that

1. $D(A)$ is stable under limits and colimits
2. $\forall M \in D(A^{\triangleright}), N \in D(A)$ we have $R\mathcal{H}om_{A^{\triangleright}}(M, N) \in D(A)$
3. Let $- \otimes_{A^{\triangleright}} A$ be the left adjoint to $D(A) \subset D(A^{\triangleright})$. Then $- \otimes_{A^{\triangleright}} A$ sends $D(A^{\triangleright})_{\geqslant 0}$ to $D(A)_{\geqslant 0}$.

Remark 2.15. One should think of $D(A^{\triangleright})$ as the category of all topological modules on the topological ring $A^{\triangleright}$, and of $D(A)$ as the category of complete modules for the analytic structure. But this analogy isn't perfect and can be slightly misleading: e.g. complete objects are stable under colimits.

We denote by AnRing the category of analytic rings. A morphism $(A^{\triangleright}, D(A)) \to (B^{\triangleright}, D(B))$ is a morphism $A^{\triangleright} \to B^{\triangleright}$ such that any $M \in D(B)$ seen as an $A^{\triangleright}$-module is in $D(A)$.

Example 2.16. Given a condensed ring $A^{\triangleright}$, one has the trivial analytic structure $A^{\triangleright}_{\text{triv}} = (A^{\triangleright}, D(A^{\triangleright}))$.

3 Introduction to Arakelov geometry, Part 2 - by Jean-Benoît Bost

We have already seen the definition of Hermitian bundles. There are two real-valued invariants in Arakelov geometry. One of them is the *Arakelov degree*, which we aim to introduce in a first place.
Let us fix the setting until the end of this section. Let K be a number field, id est $[K : \mathbb{Q}] < \infty$. Let us denote $\mathrm{Spec}(\mathcal{O}_K)$ by $\mathcal{X}$.

3.1 Real-valued invariants produced by Arakelov geometry

3.1.1 Arakelov degree

Notation. *We denote by* $Z_0(\mathcal{X})$ *the space of* 0*-cycles on* $\mathcal{X}$*, id est cycles of dimension* 0*. An element of* $Z_0(\mathcal{X})$ *can be written as a formal sum of points* $\sum_{i\in I} n_i P_i$ *where the* P_i*'s are closed points, the* n_i*'s are integers and* I *is finite.*

Notation. *By convention, we write* p *for a prime number and* $\mathfrak{p}$ *for a non-zero prime ideal of* $\mathcal{O}_K$*.*

Definition 3.1. We introduce:

$$\hat{Z}_0(\mathcal{X}) = \left(Z_0(\mathcal{X}) \oplus \mathbb{R}^{\mathcal{X}(\mathbb{C})}\right)^{c.c.} = Z_0(\mathcal{X}) \oplus \left(\mathbb{R}^{\mathcal{X}(\mathbb{C})}\right)^{c.c.}$$

The exponent *c.c.* means that we consider the invariant part under complex conjugation.

Definition 3.2. We define the map

$$\widehat{\mathrm{div}} : \begin{cases} K^\times & \to & \hat{Z}_0(\mathcal{X}) \\ q & \mapsto & \left(\sum_{\mathfrak{p}\neq 0 \text{ prime in } \mathcal{O}_K} v_{\mathfrak{p}}(q)[\mathfrak{p}], (-\log(|\sigma(q)|)_{\sigma:K\hookrightarrow\mathbb{C}})\right). \end{cases}$$

Definition 3.3. Let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime ideal. We set $\mathbb{F}_{\mathfrak{p}}$ to be $\mathcal{O}_K/\mathfrak{p}$ and $N_{\mathfrak{p}}$ to be the cardinal of $\mathbb{F}_{\mathfrak{p}}$.

Definition 3.4. We define the map

$$\widehat{\deg} : \begin{cases} \hat{Z}_0(\mathcal{X}) & \to & \mathbb{R} \\ \left(\sum_{\mathfrak{p}\neq 0 \text{ prime in } \mathcal{O}_K} n_{\mathfrak{p}}[\mathfrak{p}], (\lambda_\sigma)_{\sigma:K\hookrightarrow\mathbb{C}}\right) & \mapsto & \sum_{\mathfrak{p}\neq 0 \text{ prime in } \mathcal{O}_K} n_{\mathfrak{p}} \log(N_{\mathfrak{p}}) \end{cases}$$

where by definition $\lambda_\sigma = \lambda_{\overline{\sigma}}$.

Proposition 3.5. *The following composition*

$$K^\times \xrightarrow{\widehat{\mathrm{div}}} \hat{Z}_0(\mathcal{X}) \xrightarrow{\widehat{\deg}} \mathbb{R}$$

is zero.
In other words, we have the following formula:

$$\forall q \in K^\times,\ \widehat{\deg} \circ \widehat{\mathrm{div}}(q) = 0$$

which is named the product formula.

Proof. First, let us remark that the following composition

$$\mathbb{Q}^\times \xrightarrow{\widehat{\mathrm{div}}} \hat{Z}_0(\mathrm{Spec}(\mathbb{Z})) \xrightarrow{\widehat{\deg}} \mathbb{R}$$

is zero. Indeed, let us take some q in $\mathbb{Q}^\times$ and let us compute $\widehat{\deg} \circ \widehat{\mathrm{div}}(q)$. We have:

$$\widehat{\deg} \circ \widehat{\mathrm{div}}(q) = \widehat{\deg}\left(\sum_{p\,\mathrm{prime}} v_p(q)[p] - \log(|q|)\right) = \sum_{p\,\mathrm{prime}} v_p(q)\log(p) - \log(|q|) = 0.$$

Now, let us denote by $\pi : \mathrm{Spec}(\mathcal{O}_K) \to \mathrm{Spec}(\mathbb{Z})$ the structure morphism. Then, we consider its pushforward as follows:

$$\pi_* : \begin{cases} \hat{Z}_0(\mathrm{Spec}(\mathcal{O}_K)) \to \hat{Z}_0(\mathrm{Spec}(\mathbb{Z})) \\ \left(\sum_{\mathfrak{p}\neq 0\ \mathrm{prime\,in}\,\mathcal{O}_K} n_{\mathfrak{p}}[\mathfrak{p}], (\lambda_\sigma)_{\sigma : K \hookrightarrow \mathbb{C}}\right) \mapsto \left(\sum_{p\,\mathrm{prime}} \left(\sum_{\mathfrak{p}|p} n_{\mathfrak{p}} f_{\mathfrak{p}}\right)[p], \sum_{\sigma: K\hookrightarrow\mathbb{C}} \lambda_\sigma\right) \end{cases}$$

where the $f_{\mathfrak{p}}$'s can be computed as some residues.

Hence, we obtain a commutative diagram:

$$\begin{tikzcd}
\mathbb{Q}^\times \arrow[r, "\widehat{\mathrm{div}}"] & \hat{Z}_0(\mathrm{Spec}(\mathbb{Z})) \arrow[r, "\widehat{\deg}"] & \mathbb{R} \\
K^\times \arrow[u, "N_{K/\mathbb{Q}}"] \arrow[r, "\widehat{\mathrm{div}}"] & \hat{Z}_0(\mathcal{X}) \arrow[u, "\pi_*"] \arrow[r, "\widehat{\deg}"] & \mathbb{R}. \arrow[u, equal]
\end{tikzcd}$$

It concludes the proof. ☐

Corollary 3.6. *We can factorize the Arakelov degree map as follows:*

$$\widehat{\deg} : \hat{Z}_0(\mathcal{X})/\mathrm{div}K^\times \to \mathbb{R}$$

where we have

$$\hat{Z}_0(\mathcal{X})/\mathrm{div}K^\times \cong \widehat{\mathrm{Pic}}(\mathcal{X}).$$

Example 3.7. Let $\overline{L} := \mathcal{O}_{\mathcal{X}}(D, g)$ be a Hermitian line bundle over $\mathrm{Spec}(\mathcal{O}_K)$. We have

$$\widehat{\deg}(D, g) = \widehat{\deg}(\overline{L}) \in \mathbb{R}.$$

Example 3.8. Let $\overline{E}$ be a Hermitian vector bundle of rank N over $\mathrm{Spec}(\mathcal{O}_K)$. Then, we have

$$\widehat{\deg}(\overline{E}) = \widehat{\deg}(\bigwedge^N \overline{E}).$$

Example 3.9. Let $\overline{E} = (E, \|\cdot\|)$ a Hermitian vector bundle over $\mathrm{Spec}(\mathbb{Z})$. Then, we have

$$\widehat{\deg}(\overline{E}) = -\log(\mathrm{covol}(\overline{E})).$$

Example 3.10. We consider the line bundle $\mathcal{O}(1)$ over $\mathbb{P}^N_{\mathbb{Z}}$. We want to define a norm on $\mathcal{O}(1)_{\mathbb{C}}$ to obtain a Hermitian bundle. To do so, we consider the exact sequence

$$\mathcal{O}^{\oplus(N+1)}_{\mathbb{P}^{N+1}_{\mathbb{Z}}} \to \mathcal{O}(1) \to 0.$$

After tensorizing by $\mathbb{C}$, we obtain

$$\mathcal{O}^{\oplus(N+1)}_{\mathbb{P}^{N}_{\mathbb{C}}} \to \mathcal{O}(1)_{\mathbb{C}} \to 0.$$

We consider the standard metric on $\mathcal{O}^{\oplus(N+1)}_{\mathbb{P}^{N}_{\mathbb{C}}}$ which is given by $\sum_{i=0}^N |t_i|^2$ on each fiber. We set $\|\cdot\|$ to be the quotient metric of this standard metric. We obtain a Hermitian bundle $\overline{\mathcal{O}(1)} = (\mathcal{O}(1), \|\cdot\|)$.

Let $P \in \mathbb{P}^N_{\mathbb{Z}}(\mathbb{Q})$. Knowing that $\mathbb{P}^N_{\mathbb{Z}}(\mathbb{Q}) \cong \mathbb{P}^N_{\mathbb{Z}}(\mathbb{Z})$, we can see P as a morphism $P : \mathrm{Spec}(\mathbb{Z}) \to \mathbb{P}^N_{\mathbb{Z}}$. Then, we have

$$\widehat{\deg}(P^* \overline{\mathcal{O}(1)}) := \log \left(\sum_{i=0}^{N} x_i^2 \right)$$

where $P = (x_0 : \ldots : x_N)$.

Remark 3.11. Let us generalize the statement of Example 3.10. Let us suppose $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ to be separated and of finite type. Let us recall that we have introduced the set of 0-cycles $Z_0(\mathcal{X})$ in the beginning of Section 3.1.1. Then we have

$$\widehat{\deg} : \begin{cases} Z_0(\mathcal{X}) & \to & \mathbb{R} \\ \sum_{i \in I} n_i P_i & \mapsto & \sum_{i \in I} n_i \log(|K(P_i)|). \end{cases}$$

Fact 3.12. Let C be a closed integral subscheme of $\mathcal{X}$ of dimension 1 which is proper over $\mathrm{Spec}(\mathbb{Z})$. Let $f \in K(C)^\times$ be a non-zero meromorphic function on C. If $x \in C(\mathbb{C})$ is a complex point of C, let us denote by $f_x \in \mathbb{C}^\times$ the value of f at x. Then, we have

$$\widehat{\deg} \circ \mathrm{div} f + \sum_{x \in C(\mathbb{C})} \log(|f_x|^{-1}) = 0.$$

When $C \to \mathrm{Spec}(\mathbb{Z})$ is dominant, it can be compared with the product formula (Proposition 3.5). Otherwise, there is also an archimedean contribution with the sum over complex points of C.

Definition/Proposition 3.13. *We keep notation of Fact 3.12. Let $\overline{L}$ be a Hermitian bundle over $\mathcal{X}$ and let us fix a section $s \in \Gamma(C_\eta, L_\eta) \setminus \{0\}$, where η is the generic point. The number*

$$\widehat{\deg}(\overline{L}|C) = \mathrm{ht}_{\overline{L}} := (\widehat{\deg} \circ \mathrm{div})(s) + \sum_{x \in C(\mathbb{C})} \log(\|s(x)\|^{-1}) \in \mathbb{R}$$

is a well-defined real number. It is named the degree of $\overline{L}$ along C.

3.1.2 Arakelov intersection number on arithmetic surfaces

Definition 3.14. An arithmetic surface is an integral scheme $\mathcal{X}$ of finite type over Spec($\mathbb{Z}$) which we suppose to be flat, regular and of absolute dimension 2.

Remark 3.15. In this situation, $\mathcal{X}(\mathbb{C})$ is a Riemann surface which is compact if and only if $\mathcal{X}$ is projective over $\mathbb{Q}$.

Definition 3.16. Let $\mathcal{X}$ be an arithmetic surface and let $\overline{L} = (L, \|\cdot\|)$ be a Hermitian bundle over $\mathcal{X}$ where $\|\cdot\|$ is a $\mathcal{C}^\infty$-metric. Then we define

$$c_1(\overline{L}_\mathbb{C}) = \frac{1}{2\pi i}\partial\overline{\partial}\log(\|s\|^2)$$

where s is a locally $\mathbb{C}$-analytic section of $L_\mathbb{C}$ over the Riemann surface $\mathcal{X}(\mathbb{C})$.

Remark 3.17. We keep notation of Definition 3.16. The number $c_1(\overline{L}_\mathbb{C})$ is a well-defined $(1,1)$-form on $\mathcal{X}(\mathbb{C})$.

Remark 3.18. Let $\mathcal{X}$ be an arithmetic surface and let $\overline{L}$ be $\mathcal{O}_\mathcal{X}(D, g)$ where D is div$s$ for s a non-zero meromorphic section of L and g is $-\log(\|s\|)$. Then we have

$$c_1(\overline{L}_\mathbb{C}) = \frac{i}{\pi}\partial\overline{\partial}g + \delta_{D_\mathbb{C}} := \omega(g).$$

Definition 3.19 (Arakelov intersection number)**.** Let $\mathcal{X}$ be an arithmetic surface and let $\overline{L}$ be as in Definition 3.16. Let (D, g) be an Arakelov divisor in $\mathcal{X}$. We suppose that the support $|D|$ of D is proper over Spec($\mathbb{Z}$) and that the support of g is compact. We define

$$\overline{L}\cdot(D,g) := \widehat{\deg}(\overline{L}|C) + \int_{\mathcal{X}(\mathbb{C})} g\, c_1(\overline{L}_\mathbb{C}) \in \mathbb{R}.$$

Remark 3.20. In Definition 3.19, the integral takes into account archimedean places of (D, g) which are forgotten in $\widehat{\deg}(\overline{L}|C)$.

Definition 3.21. Let (D, g) be an Arakelov divisor as in Definition 3.19 and (D', g') be an element of $\hat{Z}_0(\mathcal{X})$ (defined in Definition 3.1). We define

$$(D', g')\cdot(D, g) := \overline{\mathcal{O}_\mathcal{X}}(D', g')\cdot(D, g).$$

3.1.3 Analysis on Riemann surfaces

In this subsection, let M be the Riemann surface $\mathcal{X}(\mathbb{C})$.

Definition 3.22 (Dirichlet product)**.** Let f_1, f_2 be functions in $\mathcal{C}^\infty(M, \mathbb{R})$. We define the Dirichlet product as follow

$$< f_1, f_2 >_{Dir} = i\int_M \partial f_1 \wedge \overline{\partial} f_2 = \frac{1}{2}\int_M \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial y}dxdy = -\frac{1}{2}\int_M f_1 \Delta f_2$$

where we write $z = x + iy$.

Remark 3.23. This product is symmetric when M is compact, id est when $\mathcal{X}$ is projective over Spec($\mathbb{Z}$). The positivity of the product is straightforward. For simplicity, let us suppose that M is compact from now on.

Definition 3.24 (Star product)**.** Let g_1 and g_2 be Green functions respectively associated to some divisors D_1 and D_2. For $\alpha \in \{1,2\}$, let us recall that we have defined a $\mathcal{C}^\infty$ $(1,1)$-form $\omega(g_\alpha)$ on M as follows

$$\omega(g_\alpha) = \frac{i}{\pi}\partial\overline{\partial}g_\alpha + \delta_{D_\alpha}$$

in Remark 3.18. Let us suppose that supports of D_1 and D_2 are disjoint. We define

$$g_1 * g_2 := g_2\,\delta_{D_1} + g_1\,\omega(g_2).$$

Remark 3.25. We keep notation of Definition 3.24. We remark that $\int_M g_1 * g_2$ is a real number.

Proposition 3.26. *We keep notation of Definition 3.24.*

1. *We have the equality*

$$\int_M g_1 * g_2 = \int_M g_2 * g_1.$$

2. *If the supports of g_1 and g_2 are disjoint, then*

$$\int_M g_1 * g_2 = 0.$$

3. *Let $f_1, f_2 \in \mathcal{C}^\infty(M,\mathbb{R})$. We have the equality*

$$\int_M (g_1+f_1)*(g_2+f_2) = \int_M g_1*g_2+\int_M (f_1\,\omega(g_2)+f_2\,\omega(g_1))-\frac{1}{\pi} < f_1, f_2 >_{Dir}$$

where $\omega(g_i)$ and $< \cdot,\cdot >_{Dir}$ are defined respectively in Remark 3.18 and in Definition 3.22.

Remark 3.27. To prove Proposition 3.26(1), one has to apply the Green formula.

Let us give a few basic facts.

Properties 3.28. *We keep notation of Definition 3.24 and of Proposition 3.26.*

1. *We have the equality*

$$(D_1, g_1+f_1)\cdot(D_2, g_2+f_2) = (D_1, g_1)\cdot(D_2, g_2)+\int_M ((f_1)\,\omega(g_2)+f_2\,\omega(g_1))-\frac{1}{\pi} < f_1, f_2 >_{Dir} .$$

2. *Let us suppose that supports of $(D_1)_{\mathbb{Q}}$ and $(D_2)_{\mathbb{Q}}$ are disjoint. Let us write $D_1 \cdot D_2$ the set of 0-cycles on the intersection of the supports of the divisors $|D_1| \cap |D_2|$. We suppose $D_1 \cdot D_2$ to be proper over $\mathbb{Z}$. Then, we have the equality*

   $$(D_1, g_1) \cdot (D_2, g_2) = \widehat{\deg}(D_1 \cdot D_2) + \int_M g_1 * g_2.$$

3. *The equality*

   $$(D_1, g_1) \cdot (D_2, g_2) = (D_2, g_2) \cdot (D_1, g_1)$$

   holds.

Remark 3.29. This setting is interesting when the fiber X_Δ over Δ is a little complicated, for example singular.

3.2 What is Arakelov geometry good for ?

3.2.1 Diophantine results (not surprising)

Arakelov geometry can be very useful to understand classical transcendence results, even though it is not necessary to do the proofs. Let us explain some links with arithmetic geometry.

Proposition 3.30. *Let us consider formal series $f(X) = \sum_{n \in \mathbb{N}} a_n X^n$ where we suppose the a_n's to be integers (or more generally we can suppose that $n!\, a_n$ is an integer) for all n in $\mathbb{N}$. Let us suppose that f is meromorphic on $\mathbb{C}$ (or more generally that we have a growth estimate at ∞). If f is non-algebraic, then (almost all) special values $f(\alpha)$ for α in $\overline{Q} \smallsetminus \{i\}$ are in $\mathbb{C} \smallsetminus \overline{\mathbb{Q}}$.*

We can state the contrapositive of Proposition 3.30 as follows.

Corollary 3.31. *Let us consider the same setting as in Proposition 3.30 but without assuming that f is non-algebraic. If many values $f(\alpha)$ are in $\mathbb{Q}$, then f is algebraic.*

Remark 3.32. A proof of Proposition 3.30 mimics proofs of algebraic geometry, see GAGA, Grothendieck, Hartshorne...

Remark 3.33. Arakelov geometry also allows to do some mixes between algebraic geometry and arithmetic geometry. It can seem strange that we manage to obtain results which do not depend on the chosen model. For example, let us consider X a variety over $\mathbb{Q}$ and we consider a model $\mathcal{X}$ and a Hermitian line bundle $\overline{L}$ over $\mathcal{X}$. Then the results depend on the asymptotic $\overline{L}^{\otimes n}$ for n very big. This asymptotic does not depend on the choice of the model.

3.2.2 Ineffective proof of finiteness of rational points (very surprising)

Theorem 3.34 (Mordell)**.** *Let C be a projective curve over $\mathbb{Q}$. Then $C(\mathbb{Q})$ is finite. If K is a number field, $C(K)$ is finite.*

Remark 3.35. Let us consider $C \hookrightarrow \mathbb{P}^N_{\mathbb{Q}}$. Instead of considering points in $C(\mathbb{Q})$, the proof considers pairs $(P, Q) \in C(\mathbb{Q})^2$. Hence, it does not say anything if $|C(\mathbb{Q})| = 1$. Otherwise, there is some control over $\mathrm{ht}(P)/\mathrm{ht}(Q)$.
For p, q, n integers, we define the line bundle

$$M(p,q,n) := (\mathcal{O}(p) \otimes \mathcal{O}(q))(n\Delta)$$

where we twist n times by the diagonal. Then

$$\mathrm{ht}_{\overline{M(p,q,n)}}(P,Q) = p\,\mathrm{ht}(P) + q\,\mathrm{ht}(Q) + n\,\widehat{\deg}(\overline{P} \cdot \overline{Q}) + n\,\log(d_\infty(P,Q)^{-1}).$$

4 Analytic rings II - by Arthur-César Le Bras

We denote by AnRing the category of analytic rings. Yesterday we formulated the definition of analytic rings in a way which is the right one when dealing with general animated ("derived") condensed rings. It is important to build the theory correctly to work in the derived framework, but for today at least we will only consider static (underived) condensed rings. In this case, let us now give a different point of view on the notion of analytic ring structure.

Definition 4.1. Let $A^{\triangleright}$ be a condensed ring. An analytic ring structure A on $A^{\triangleright}$ is the datum of a full abelian subcategory $\mathrm{Mod}_A \subset \mathrm{Mod}_{A^{\triangleright}}$ such that

1. it is stable under limits, colimits and extensions.
2. it contains $A^{\triangleright}$ (i.e. the ring itself is "complete").
3. for all $M \in \mathrm{Mod}_{A^{\triangleright}}$ and $N \in \mathrm{Mod}_A$ and for all i we have $\underline{\mathrm{Ext}}^i(M, N) \in \mathrm{Mod}_A$.

This is a slightly different definition from the one we gave yesterday, see [3, Lecture 13/24] for the precise relation between the two definitions. If we have Mod_A as in Definition 4.1, then $\mathrm{D}(A) \coloneqq \{M \in \mathrm{D}(A^{\triangleright})$ s.t. $\mathrm{H}^q(M) \in \mathrm{Mod}_A$ for all $q\}$ defines an analytic ring structure on $A^{\triangleright}$ in the sense of Definition 2.14. Conversely, if we have such a category $\mathrm{D}(A)$, the abelian category $\mathrm{Mod}_A \coloneqq \mathrm{D}(A) \cap \mathrm{Mod}_{A^{\triangleright}}$ defines an analytic ring structure in the sense of Definition 4.1. For today, we stick with this abelian definition.

Notation 4.2. If $A = (A^{\triangleright}, \mathrm{Mod}_A)$ is an analytic ring and $S \in \mathrm{Prof}^{\mathrm{light}}$, we set $A[S] \coloneqq A^{\triangleright}[S] \otimes^L_{A^{\triangleright}} A$.

Definition 4.3. Let $A = (A^{\triangleright}, \mathrm{Mod}_A)$ and $B = (B^{\triangleright}, \mathrm{Mod}_B)$ be analytic rings. A morphism of analytic rings $A \to B$ is a morphism of condensed rings $A^{\triangleright} \to B^{\triangleright}$ such that for all $M \in \mathrm{Mod}_B$, we have $M \in \mathrm{Mod}_A$.

The goal of today's talk is to give concrete interesting examples of analytic rings. Before doing that, we state the following result

Proposition 4.4. *The category* Mod_A *has a unique symmetric monoidal structure commuting with colimits in each variable and making* $- \otimes_{A^{\triangleright}} A$ *symmetric monoidal. Moreover, the functor*

$$\mathrm{Mod}_{A^{\triangleright}} \xrightarrow{-\otimes_{A^{\triangleright}} B^{\triangleright}} \mathrm{Mod}_{B^{\triangleright}} \xrightarrow{-\otimes_{B^{\triangleright}} B} \mathrm{Mod}_B$$

factors uniquely over

$$\mathrm{Mod}_{A^{\triangleright}} \xrightarrow{-\otimes_{A^{\triangleright}} A} \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B.$$

The same statement holds on the derived level.

Notation 4.5. We denote $-\otimes_A B : \mathrm{Mod}_A \to \mathrm{Mod}_B$, resp. $-\otimes^L_A B : \mathrm{D}(A) \to \mathrm{D}(B)$ this unique factorisation.

Proof. Exercise. The key point is that the kernel K of $-\otimes_{A^\triangleright} A$ is a tensor ideal, meaning that if $M \in \mathrm{Mod}_A$ is killed by $-\otimes_{A^\triangleright} A$, the same holds for $M \otimes_{A^\triangleright} N$ for all $N \in \mathrm{Mod}_{A^\triangleright}$. ☐

Example 4.6 (Trivial analytic ring structure)**.** Let $A^\triangleright$ be a condensed ring. Then $(A^\triangleright, \mathrm{Mod}_{A^\triangleright})$ is an analytic ring. This trivial example is already interesting, even when $A^\triangleright$ is discrete: indeed, $\mathrm{Mod}_{A^\triangleright}$ is the category of condensed $A^\triangleright$-modules and extends the usual category of *discrete* $A^\triangleright$-modules. This will allow us to define a six functor formalism on quasicoherent sheaves of schemes.

Example 4.7 (The induced analytic ring structure)**.** Let $A = (A^\triangleright, \mathrm{Mod}_A)$ be an analytic ring and let $B^\triangleright$ be an $A^\triangleright$-algebra. Assume moreover that $B^\triangleright \in \mathrm{Mod}_A$. We define

$$\mathrm{Mod}_{B^\triangleright_{/A}} \coloneqq \{M \in \mathrm{Mod}_{B^\triangleright} \text{ s.t. } M \in \mathrm{Mod}_A \text{ when seen as } A^\triangleright\text{-module}\}.$$

Then $B^\triangleright_{/A} \coloneqq (B^\triangleright, \mathrm{Mod}_{B^\triangleright_{/A}})$ defines an analytic ring structure on $B^\triangleright$, called the *induced analytic ring structure*. This example is useful because if we build interesting analytic ring structures on $\mathbb{Z}$, it allows us to obtain analytic ring structures on bigger rings without further work.

4.1 The solid analytic ring structure

We introduce a non-trivial analytic ring structure $\mathbb{Z}_\square$ on $\mathbb{Z}$ which is relevant to non-archimedean analysis, a place where a sequence is summable if and only if it goes to 0. To define this analytic ring structure, the idea is to translate this condition on null-sequences in condensed terms thanks to the light profinite set $\mathbb{N} \cup \{\infty\}$. We set $P \coloneqq \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/\mathbb{Z}[\infty] \in \mathrm{Cond}(\mathrm{Ab})$. Then for all $X \in \mathrm{Cond}(\mathrm{Ab})$, the abelian group $\mathrm{Hom}(P, X)$ can be intuitively thought as the abelian group of null-sequences in X.

Remark 4.8 (Warning)**.** This is just an intuition. Even when X comes from a topological abelian group, $\mathrm{Hom}(P, X)$ does not always coincide with convergent null-sequences. Indeed, if X is not separated a sequence can have different limits: in other words, the morphism $\mathrm{Hom}(P, X) \to \mathrm{Hom}(\mathbb{Z}[\mathbb{N}], X)$ may not be injective.

Notation 4.9. P actually has an algebra structure: when we see it as an algebra, we denote it by $\mathbb{Z}[\hat{X}]$. This notation is justified by the fact that $\mathbb{Z}[\mathbb{N}]$ seen as an algebra naturally identifies with the ring of polynomials $\mathbb{Z}[X]$, via the morphism $[n] \mapsto X^n$.

Let $\mathrm{shift} : P \to P$ be the endomorphism of P induced by $[n] \mapsto [n+1]$.

Definition 4.10. Let $M \in \mathrm{Cond}(\mathrm{Ab})$. We say that M is *solid* if $\underline{\mathrm{Hom}}(P, M) \xrightarrow{\text{id}-\text{shift}} \underline{\mathrm{Hom}}(P, M)$ is an isomorphism. We call Solid the correspondent subcategory of $\mathrm{Mod}_{\mathbb{Z}} = \mathrm{Cond}(\mathrm{Ab})$.

What is the intuition behind this definition? The morphism $\mathrm{id} - \mathrm{shift} : \underline{\mathrm{Hom}}(P, M) \to \underline{\mathrm{Hom}}(P, M)$ can be thought in terms of null-sequences as

$$\{\text{null-sequences in } M\} \to \{\text{null-sequences in } M\}, \qquad (x_n)_{n\in\mathbb{N}} \mapsto (x_n - x_{n+1})_{n\in\mathbb{N}}.$$

The inverse of this map, if it exists, should be

$$\{\text{null-sequences in } M\} \to \{\text{null-sequences in } M\}, \qquad (x_n)_{n\in\mathbb{N}} \mapsto (\sum_{i=0}^{\infty} x_i, \sum_{i=1}^{\infty} x_i, \sum_{i=2}^{\infty} x_i, \dots).$$

Consequently, saying that this inverse exists (i.e. saying that $\mathrm{id}-\mathrm{shift} : \underline{\mathrm{Hom}}(P, M) \to \underline{\mathrm{Hom}}(P, M)$ is an isomorphism) amounts to say that every null-sequence is summable.

Theorem 4.11. $\mathbb{Z}_\square \coloneqq (\mathbb{Z}, \mathrm{Solid})$ *is an analytic ring structure on* $\mathbb{Z}$.

Proof. Stability by limits, colimits and extensions follows from internal projectivity and compactness of P as an object of $\mathrm{Cond}(\mathrm{Ab})$.

Let us prove the stability under $\underline{\mathrm{Ext}}^i$. Let M be a condensed abelian group and N be a solid abelian group. By internal projectivity of P, we have

$$\underline{\mathrm{Hom}}(P, \underline{\mathrm{Ext}}^i(M, N)) = \underline{\mathrm{Ext}}^i(P \otimes M, N),$$

which canonically identifies with $\underline{\mathrm{Ext}}^i(M, \underline{\mathrm{Hom}}(P, N))$. Since N is solid, id−shift is an automorphism of $\underline{\mathrm{Hom}}(P, N)$. Thus the same holds for $\underline{\mathrm{Hom}}(P, \underline{\mathrm{Ext}}^i(M, N)) = \underline{\mathrm{Ext}}^i(M, \underline{\mathrm{Hom}}(P, N))$.

The only thing left to show is that $\mathbb{Z}$ is solid. We have an identification $\underline{\mathrm{Hom}}(P, \mathbb{Z}) \simeq \bigoplus_{n\in\mathbb{N}} \mathbb{Z}$, under which the morphism $\mathrm{id} - \mathrm{shift}$ becomes

$$\bigoplus_{n\in\mathbb{N}} \mathbb{Z} \longrightarrow \bigoplus_{n\in\mathbb{N}} \mathbb{Z}, \qquad (x_n)_{n\in\mathbb{N}} \mapsto (x_n - x_{n+1})_{n\in\mathbb{N}}.$$

This is an isomorphism with inverse $(x_n)_{n\in\mathbb{N}} \mapsto (\sum_{i=n}^{\infty} x_i)_{n\in\mathbb{N}}$. (This is not surprising: $\mathbb{Z}$ is discrete, so every null-sequence is eventually constant, hence summable). □

Notation. We set $(-)^\square \coloneqq - \otimes_{\mathbb{Z}} \mathbb{Z}_\square$ and we call it solidification.

With the light formalism, the proof has become easy because of the internal projectivity of P in $\mathrm{Cond}(\mathrm{Ab})$. What is less clear from this formalism is how to compute $\mathbb{Z}[S]^\square$ for $S \in \mathrm{Prof}^{\mathrm{light}}$. We have the following (with no proof)

Fact 4.12. Let $S = \lim_i S_i$ be an infinite light profinite set. Then we have

$$\mathbb{Z}_\square[S] \coloneqq \mathbb{Z}[S]^{L\square} = \lim_i \mathbb{Z}[S_i] \simeq \prod_{\mathbb{N}} \mathbb{Z}.$$

In particular, $\prod_{\mathbb{N}} \mathbb{Z}$ is a compact projective generator of Solid.

To see the last isomorphism, observe that since $\mathbb{Z}$ is discrete, every continuous map $S \to \mathbb{Z}$ factors through a finite quotient, thus we have $\mathcal{C}(S, \mathbb{Z}) = \operatorname{colim}_i \mathcal{C}(S_i, \mathbb{Z})$ and thus

$$\lim_i \mathbb{Z}[S_i] = \underline{\mathrm{Hom}}(\mathcal{C}(S, \mathbb{Z}), \mathbb{Z}).$$

Moreover, one can prove that $\mathcal{C}(S, \mathbb{Z})$ is a free abelian group. The choice of a basis gives isomorphisms $\mathcal{C}(S, \mathbb{Z}) \simeq \bigoplus_{\mathbb{N}} \mathbb{Z}$ and $\lim_i \mathbb{Z}[S_i] \simeq \prod_{\mathbb{N}} \mathbb{Z}$.

Beware that in general $(-)^{L\square}$ is really derived, as the following remark shows

Remark 4.13. Let X be a CW complex. Then we have

$$\mathbb{Z}[X]^{L\square} \simeq C_\bullet(X, \mathbb{Z}),$$

where $C_\bullet(X, \mathbb{Z})$ denotes the complex of singular chains. This result (which is a bit surprising) implies that $(-)^{L\square}$ can have a contribution in arbitrary negative cohomological degree. To prove this, one observes that we have

$$R\Gamma_{\mathrm{sing}}(X, \mathbb{Z}) = R\underline{\mathrm{Hom}}(\mathbb{Z}[X], \mathbb{Z}) = R\underline{\mathrm{Hom}}(\mathbb{Z}[X]^{L\square}, \mathbb{Z}),$$

where the last equality uses the fact that $\mathbb{Z}$ is solid. On the other hand, $R\Gamma_{\mathrm{sing}}(X, \mathbb{Z})$ has $C_\bullet(X, \mathbb{Z})$ as $\mathbb{Z}$-linear dual. Thus we get an induced map

$$\mathbb{Z}[X]^{L\square} \to C_\bullet(X, \mathbb{Z})$$

and we need to check it is an isomorphism, which can be done by descent from the case where X is a profinite set.

Notation 4.14. We set $- \otimes_{\mathbb{Z}}^{L\square} - := - \otimes_{\mathbb{Z}_\square}^{L} -$.

Remark 4.15 (Solid tensor product computations)**.** We have the following solid tensor product computations

$$1) \qquad \prod_{\mathbb{N}} \mathbb{Z} \otimes_{\mathbb{Z}}^{L\square} \prod_{\mathbb{N}} \mathbb{Z} \simeq \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}.$$

Proof. Using the isomorphism $\mathbb{N} \times \mathbb{N} \xrightarrow{\sim} \mathbb{N}$ and the fact that $\mathbb{Z}[-]$ is symmetric monoidal, we have $P \otimes_{\mathbb{Z}} P \simeq P$. If we combine this with the isomorphism $P^{L\square} \simeq \prod_{\mathbb{N}} \mathbb{Z}$ we get the desired computation. □

From this we get many other computations:

$$2) \qquad \mathbb{Z}[\![t]\!] \otimes_{\mathbb{Z}}^{L\square} \mathbb{Z}[\![u]\!] \simeq \mathbb{Z}[\![t, u]\!].$$

Proof. As a condensed abelian group, we have $\mathbb{Z}[\![t]\!] \simeq \prod_{\mathbb{N}} \mathbb{Z}$. Hence this computation is a direct consequence of 1), where we also take care of the ring structure of $\mathbb{Z}[\![t]\!]$, $\mathbb{Z}[\![u]\!]$ and $\mathbb{Z}[\![t, u]\!]$. □

$$3) \qquad \mathbb{Z}_p \otimes_{\mathbb{Z}}^{L\square} \mathbb{Z}_\ell \simeq \begin{cases} \mathbb{Z}_p & \text{if } p = \ell, \\ 0 & \text{if } p \neq \ell. \end{cases}$$

Proof. The trick is to write $\mathbb{Z}_p = \mathbb{Z}[\![t]\!]/(t-p)$ and similarly for $\mathbb{Z}_\ell$ and to use 2). ☐

This computation tells us that the solid tensor product has better properties than $\otimes_{\mathbb{Z}}^L$: indeed, the underlying abelian group of $\mathbb{Z}_p \otimes_{\mathbb{Z}}^L \mathbb{Z}_\ell$ would just be the algebraic tensor product, which is huge.

$$4) \qquad (\bigoplus_{\mathbb{N}} \mathbb{Z}_p)^{\wedge p}[\frac{1}{p}] \otimes_{\mathbb{Z}}^{L\square} (\bigoplus_{\mathbb{N}} \mathbb{Z}_p)^{\wedge p}[\frac{1}{p}] = (\bigoplus_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}_p)^{\wedge p}[\frac{1}{p}].$$

Proof. To compute solid tensor products we have two tools: the basic computations 1), 2), 3) and the commutation with colimits in both variables. Thus inverting p is not a problem. The difficult part is to deal with the p-adic completion, since it is not a colimit but a limit. The trick is to write the p-adic completion of $\bigoplus_{\mathbb{N}} \mathbb{Z}_p$ as a colimit as follows (see [1, Lemma 5.33])

$$(\bigoplus_{\mathbb{N}} \mathbb{Z}_p)^{\wedge p} = \bigcup_{\substack{f:\mathbb{N}\to\mathbb{N} \\ f(n)\to\infty \\ \text{if } n\to\infty}} \prod_{\mathbb{N}} p^{f(n)} \mathbb{Z}_p.$$

Thus we have

$$(\bigoplus_{\mathbb{N}} \mathbb{Z}_p)^{\wedge p}[\frac{1}{p}] \otimes_{\mathbb{Z}}^{L\square} (\bigoplus_{\mathbb{N}} \mathbb{Z}_p)^{\wedge p}[\frac{1}{p}] = ((\bigoplus_{\mathbb{N}} \mathbb{Z}_p)^{\wedge p} \otimes_{\mathbb{Z}}^{L\square} (\bigoplus_{\mathbb{N}} \mathbb{Z}_p)^{\wedge p})[\frac{1}{p}] =$$

$$= \operatorname*{colim}_{\substack{f,g:\mathbb{N}\to\mathbb{N} \\ \text{going to } \infty}} \prod_{\mathbb{N}\times\mathbb{N}} p^{f(n)+g(n)} \mathbb{Z}_p \simeq \operatorname*{colim}_{\substack{h:\mathbb{N}\times\mathbb{N}\to\mathbb{N} \\ \text{going to } \infty}} \prod_{(n,m)\in\mathbb{N}\times\mathbb{N}} p^{h(n,m)} \mathbb{Z}_p.$$

The last equality here comes from cofinality between the two filtered systems. ☐

This applies in particular to p-adic Banach spaces, i.e. $\mathbb{Q}_p$-vector spaces which are complete with respect to a non-archimedean norm. The choice of an orthonormal basis I gives an identification $V \simeq (\bigoplus_I \mathbb{Z}_p)^{\wedge p}[\frac{1}{p}]$, where $(\bigoplus_I \mathbb{Z}_p)^{\wedge p}$ is the unit ball for the norm (see [1, Lemma 5.33]). Moreover, we can assume I countable (in general, V can be written as filtered colimit of p-adic Banach spaces where I can be assumed to be countable). Thus, if we have two p-adic Banach spaces V and W, we can present them as

$$V \simeq (\bigoplus_{\mathbb{N}} \mathbb{Z}_p)^{\wedge p}[\frac{1}{p}], \qquad W \simeq (\bigoplus_{\mathbb{N}} \mathbb{Z}_p)^{\wedge p}[\frac{1}{p}]$$

and the computation $V \otimes^{L\square} W$ is given by 4). As a particular case, we get the following computation

$$\mathbb{Q}_p\langle t \rangle \otimes_{\mathbb{Z}}^{L\square} \mathbb{Q}_p\langle u \rangle \simeq \mathbb{Q}_p\langle t, u \rangle,$$

where $\mathbb{Q}_p\langle x_1, \dots, x_n\rangle$ denotes the Tate algebra in n variables. This is an important computation, since it matches with the one we use when doing p-adic analytic geometry.

4.2 Relative solid analytic ring structure

Let R be a discrete ring. Since R is solid, by Example 4.7 we can define the analytic ring

$$R_{/\mathbb{Z}_\square} = (R, \{M \in \mathrm{Mod}_R \,|\, M \text{ is solid}\}).$$

For every light profinite set S, we have the free objects

$$R_{/\mathbb{Z}_\square}[S] \coloneqq R[S] \otimes^L_R R_{/\mathbb{Z}_\square} = (\mathbb{Z}[S] \otimes^L_{\mathbb{Z}} R) \otimes^L_R R_{/\mathbb{Z}_\square} = (\mathbb{Z}[S]^{L\square}) \otimes_{\mathbb{Z}} R = (\prod_{\mathbb{N}} \mathbb{Z}) \otimes_{\mathbb{Z}} R.$$

(observe that computing $- \otimes^L R_{/\mathbb{Z}_\square}$ is the same thing as solidifying and then base-changing to R). In general, there is no reason why these free objects should coincide with $\prod_{\mathbb{N}} R$. We are now going to see another analytic ring structure on R such that the free objects are $\prod_{\mathbb{N}} R$.

Definition 4.16. Let $f \in R$. We say that $M \in \mathrm{Mod}_{R_{/\mathbb{Z}_\square}}$ is f-solid if $\mathrm{id} - f\mathrm{shift} : \underline{\mathrm{Hom}}(R \otimes_{\mathbb{Z}} P, M) \to \underline{\mathrm{Hom}}(R \otimes_{\mathbb{Z}} P, M)$ is an isomorphism.

Remark 4.17. Since R is solid, I could also have taken $R \otimes^\square_{\mathbb{Z}} P$ instead of $R \otimes_{\mathbb{Z}} P$ in the definition: it would have changed nothing.

Again, the intuition is that a solid R-module M is f-solid if every null-sequence $(x_n)_{n\in\mathbb{N}}$ is such that the sequence $(f^n x_n)_{n\in\mathbb{N}}$ is summable. Note that if $f = 1$, the condition is exactly the one of being solid, which is always satisfied by M by hypothesis.

We set $\mathrm{Mod}_{R_\square} \coloneqq \{M \in \mathrm{Mod}_{R_{/\mathbb{Z}_\square}} \,|\, \text{M is } f\text{-solid for all } f \in R\}$.

Proposition 4.18. *$R_\square \coloneqq (R, \mathrm{Mod}_{R_\square})$ is an analytic ring structure on R. Moreover, for $S = \lim_i S_i$ light profinite, we have*

$$R_\square[S] = \operatorname*{colim}_{\substack{R' \subseteq R \\ \textit{finite type} \\ \mathbb{Z}\textit{-algebra}}} \lim_i R'[S_i].$$

In particular, if R is a finite-type $\mathbb{Z}$-algebra we have $R_\square[S] = \lim_i R[S_i] \simeq \prod_{\mathbb{N}} R$.

We consider the case of the polynomial algebra $R = \mathbb{Z}[T]$: we explicit the $\mathbb{Z}[T]$-solidification functor and we show how to compute $\mathbb{Z}[T]_\square[S]$ in this case. We have

$$\mathbb{Z}[T] \otimes^\square_{\mathbb{Z}} P \simeq \mathbb{Z}[T]_{/\mathbb{Z}_\square} \otimes_{\mathbb{Z}} \mathbb{Z}[\hat{X}] \simeq \mathbb{Z}[\![X]\!][T],$$

where the last isomorphism is given by the fact that computing $- \otimes_{\mathbb{Z}} \mathbb{Z}[T]_{/\mathbb{Z}_\square}$ is the same as solidifying and base-changing to $\mathbb{Z}[T]$, plus the fact that we have $P^\square = \mathbb{Z}[\![X]\!]$. Under this identification, the shift morphism on $P^\square$ corresponds to the multiplication by X in $\mathbb{Z}[\![X]\!]$. Thus $M \in \mathrm{Mod}_{\mathbb{Z}[T]_{/\mathbb{Z}_\square}}$ is T-solid if and only if we have

$$\mathrm{id} - X \cdot T : \underline{\mathrm{Hom}}(\mathbb{Z}[\![X]\!][T], M) \to \underline{\mathrm{Hom}}(\mathbb{Z}[\![X]\!][T], M)$$

is an isomorphism.

Remark 4.19. One can show that M lies in $\mathrm{Mod}_{\mathbb{Z}[T]_\square}$ if and only if it is T-solid. More generally, if R is a finite type $\mathbb{Z}$-algebra with generators $r_1, \ldots, r_n$, M lies in $\mathrm{Mod}_{R_\square}$ if and only if it is r_i-solid for $i = 1, \ldots, n$.

We have a short exact sequence

$$0 \longrightarrow \mathbb{Z}[\![X]\!][T] \overset{\mathrm{id}-XT}{\longrightarrow} \mathbb{Z}[\![X]\!][T] \longrightarrow \mathbb{Z}((T^{-1})) \longrightarrow 0. \tag{1}$$

Consequently, for $M \in \mathrm{D}(\mathbb{Z}[T]_{/\mathbb{Z}_\square})$ (note that we are switching to the derived definition), M is T-solid if and only if we have $R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1})), M) = 0$.

We can describe the functor $- \otimes_{\mathbb{Z}[T]_{/\mathbb{Z}_\square}} \mathbb{Z}[T]_\square : \mathrm{D}(\mathbb{Z}[T]_{/\mathbb{Z}_\square}) \to \mathrm{D}(\mathbb{Z}[T]_\square)$ as follows. For $M \in \mathrm{D}(\mathbb{Z}[T]_{/\mathbb{Z}_\square})$, we have

$$M \otimes_{\mathbb{Z}[T]/\mathbb{Z}_\square} \mathbb{Z}[T]_\square \simeq R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathrm{fib}\left(\mathbb{Z}[T] \to \mathbb{Z}((T^{-1}))\right), M).$$

The key observation to deduce this is that $\mathbb{Z}((T^{-1}))$ is an idempotent algebra in $\mathrm{D}(\mathbb{Z}[T]_{/\mathbb{Z}_\square})$ (i.e. the canonical map $\mathbb{Z}((T^{-1})) \to \mathbb{Z}((T^{-1})) \otimes_{\mathbb{Z}[T]_{/\mathbb{Z}_\square}} \mathbb{Z}((T^{-1}))$ is an isomorphism). One can show that we have $\mathrm{fib}\left(\mathbb{Z}[T] \to \mathbb{Z}((T^{-1}))\right) = T^{-1}\mathbb{Z}[\![T^{-1}]\!][-1]$ (easy at the level of modules, but one needs to be careful about the algebra structure). We get an explicit description of the functor

$$-\otimes_{\mathbb{Z}[T]_{/\mathbb{Z}_\square}} \mathbb{Z}[T]_\square : \mathrm{D}(\mathbb{Z}[T]_{/\mathbb{Z}_\square}) \to \mathrm{D}(\mathbb{Z}[T]_\square), \quad M \mapsto R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(T^{-1}\mathbb{Z}[\![T^{-1}]\!][-1], M).$$

We can do even better if $M = N[T]$ for some $N \in \mathrm{D}(\mathbb{Z}_\square)$. Indeed we can rewrite it as

$$M \otimes_{\mathbb{Z}[T]_{/\mathbb{Z}_\square}} \mathbb{Z}[T]_\square \simeq R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(T^{-1}\mathbb{Z}[\![T^{-1}]\!][-1], N[T]) \simeq R\underline{\mathrm{Hom}}_{\mathbb{Z}}(u\mathbb{Z}[\![u]\!], N),$$

where the $\mathbb{Z}[T]$-module structure on $u\mathbb{Z}[\![u]\!]$ is given by $T \cdot u = 0$, $T \cdot u^2 = u$, $T \cdot u^3 = u^2$, … This formula tells us that we have

$$-[T] \otimes_{\mathbb{Z}[T]_{/\mathbb{Z}_\square}} \mathbb{Z}[T]_\square = R\underline{\mathrm{Hom}}_{\mathbb{Z}}(u\mathbb{Z}[\![u]\!], -) : \mathrm{D}(\mathbb{Z}_\square) \to \mathrm{D}(\mathbb{Z}[T]_\square).$$

Consequently, this functor commutes not only with colimits (which we already know, since it a left adjoint) but also with limits (since it is expressed as a covariant $\underline{\mathrm{Hom}}$). This property is useful to check that we have

$$\mathbb{Z}[T]_\square[S] = \lim_i \mathbb{Z}[T][S_i] = \prod_{\mathbb{N}} \mathbb{Z}[T].$$

Indeed, it allows us to reduce this to the case where S is finite.

Example 4.20. A consequence of the fact that the the functor $-[T] \otimes_{\mathbb{Z}[T]/\mathbb{Z}_{\square}} \mathbb{Z}[T]_{\square} : \mathrm{D}(\mathbb{Z}_{\square}) \to \mathrm{D}(\mathbb{Z}[T]_{\square})$ commutes with limits is that we have

$$\mathbb{Q}_p[T] \otimes_{\mathbb{Z}[T]/\mathbb{Z}_{\square}} \mathbb{Z}[T]_{\square} = \mathbb{Q}_p\langle T\rangle.$$

Remark 4.21. More geometrically, the functor $\mathrm{D}(\mathbb{Z}_{\square}) \to \mathrm{D}(\mathbb{Z}[T]_{\square})$ is the pullback functor from the affine line. The fact that it commutes with limits means that the corresponding morphism between geometric objects is smooth.

4.3 Gaseous analytic ring structure

The solid theory is not able to include archimedean objects: for example, we have $\mathbb{R}^{L\square} = 0$. This is not surprising, since being solid means that null-sequences are summable, which is false in $\mathbb{R}$. Nevertheless, if $(x_n)_{n\in\mathbb{N}}$ is a null-sequence in $\mathbb{R}$, the sequence $(r^n x_n)_{n\in\mathbb{N}}$ is summable if $r \in \mathbb{R}$ and $r < 1$.

Definition 4.22. We define $\mathrm{Mod}_{\mathbb{R}_{\mathrm{gas}}}$ as the full subcategory of $\mathrm{Mod}_{\mathbb{R}}$ of those condensed $\mathbb{R}$-modules M for which $\mathrm{id} - \frac{1}{2}\mathrm{shift} : \underline{\mathrm{Hom}}_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{Z}} P, M) \to \underline{\mathrm{Hom}}_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{Z}} P, M)$ is an isomorphism.

Remark 4.23. $r = \frac{1}{2}$ is just a choice.

This defines an analytic ring structure $\mathbb{R}_{\mathrm{gas}} := (\mathbb{R}, \mathrm{Mod}_{\mathbb{R}_{\mathrm{gas}}})$ on $\mathbb{R}$. The proof of this fact is not difficult because of internal projectivity of P.

Notation 4.24. We set $(-)^{\mathrm{gas}} := - \otimes_{\mathbb{R}} \mathbb{R}_{\mathrm{gas}} : \mathrm{Mod}_{\mathbb{R}} \to \mathrm{Mod}_{\mathbb{R}_{\mathrm{gas}}}$.

What is really non-trivial is the computation of the free objects $\mathbb{R}_{\mathrm{gas}}[S]$ for a light profinite set S and of the object $(\mathbb{R} \otimes_{\mathbb{Z}} P)^{\mathrm{gas}}$. Clausen and Scholze managed to carry out this computation for all S. We only give the result for $(\mathbb{R} \otimes_{\mathbb{Z}} P)^{\mathrm{gas}}$.

Proposition 4.25. *We set* $\mathbb{R}_{\mathrm{gas}}[\hat{X}] := (\mathbb{R} \otimes_{\mathbb{Z}} P)^{\mathrm{gas}}$. *Then we have*

$$\mathbb{R}_{\mathrm{gas}}[\hat{X}] = \{\sum_{n=0}^{\infty} a_n X^n \mid (a_n)_{n\in\mathbb{N}} \textit{ has quasi-exponential decay, i.e. } \exists \varepsilon, c > 0, 0 < a < 1 \textit{ s.t. } |a_n| \leq c a^{n^{\varepsilon}}\}$$

Idea of proof. We write a short exact sequence like (1), and get a condensed $\mathbb{R}$-module $\mathbb{R}_{\infty}$ taking the role of $\mathbb{Z}((T^{-1}))$. We obtain a characterisation of gaseous modules as those $M \in \mathrm{Mod}_{\mathbb{R}}$ such that $R\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathbb{R}_{\infty}, M) = 0$. This allows us to obtain an abstract formula for

$$(-)^{\mathrm{gas}} : \mathrm{D}(\mathrm{Mod}_{\mathbb{R}}) \to \mathrm{D}(\mathrm{Mod}_{\mathbb{R}_{\mathrm{gas}}}), \quad M \mapsto R\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathrm{fib}\big(\mathbb{R} \to \mathbb{R}_{\infty}\big), M).$$

We then need to explicit it for $M = \mathbb{R} \otimes_{\mathbb{Z}} P$ to get the desired result. □

This is a bit mysterious. The definition of $\mathbb{R}_{\mathrm{gas}}$ is quite natural but $\mathbb{R}_{\mathrm{gas}}[S]$ have a very difficult description.

The gaseous theory is particularly useful when one wants to do complex geometry. Indeed, in this case one should be allowed to do tensor products between rings of holomorphic functions, and it turns out that the gaseous tensor product gives the expected computations. Let $\mathbb{D}$ be a closed disk in $\mathbb{C}$ and let $\mathcal{O}(\mathbb{D})^{\dagger} \coloneqq \{\text{overconvergent holomorphic functions in } \mathbb{D}\}$. We can naturally see them as modules over $\mathbb{C}_{\text{gas}} \coloneqq \mathbb{C}_{/\mathbb{R}_{\text{gas}}}$.

Proposition 4.26. *We have*

$$\mathcal{O}(\mathbb{D})^{\dagger} \otimes_{\mathbb{C}_{\text{gas}}} \mathcal{O}(\mathbb{D})^{\dagger} \simeq \mathcal{O}(\mathbb{D} \times \mathbb{D})^{\dagger}.$$

This is the geometric expected property.

Idea of proof. Again, the only tools we have are the commutation with colimits in both variables and the basic computation (which here is Proposition 4.25). We can write

$$\mathcal{O}(\mathbb{D})^{\dagger} = \operatorname*{colim}_{\substack{\mathbb{D} \subset U \\ U \text{ open disk}}} \mathcal{O}(U).$$

Now we can rewrite this as a colimit of modules of the form $\mathbb{R}_{\text{gas}}[\hat{X}] \otimes_{\mathbb{R}} \mathbb{C}$. As in Remark 4.15, we use the fact that $P \otimes_{\mathbb{Z}} P = P$ to deduce the result. $\square$

4.4 Towards the gaseous base stack

This is an anticipation of an object we will see in more detail in Ferdinand's last talk. Let us replace $\mathbb{R}$ in Definition 4.22 by the ring $\mathbb{Z}[\hat{q}][q^{-1}]$. Here $\mathbb{Z}[\hat{q}]$ is an algebra isomorphic to P: we formally changed X with q to distinguish it from the copy of P that will appear in the definition of gaseous $\mathbb{Z}[\hat{q}][q^{-1}]$-modules.

Definition 4.27. A condensed $\mathbb{Z}[\hat{q}][q^{-1}]$-module $M \in \operatorname{Mod}_{\mathbb{Z}[\hat{q}][q^{-1}]}$ is *gaseous* if

$$\mathrm{id} - q \cdot \mathrm{shift} : \underline{\mathrm{Hom}}_{\mathbb{Z}[\hat{q}][q^{-1}]}(\mathbb{Z}[\hat{q}][q^{-1}] \otimes_{\mathbb{Z}} P, M) \to \underline{\mathrm{Hom}}_{\mathbb{Z}[\hat{q}][q^{-1}]}(\mathbb{Z}[\hat{q}][q^{-1}] \otimes_{\mathbb{Z}} P, M)$$

is an isomorphism.

Since $\mathbb{Z}[\hat{q}][q^{-1}]$ is not itself gaseous, this does not define an analytic ring yet (but only a *pre*-analytic ring). However, thanks to a completion procedure (in the sense of analytic rings, which is the left adjoint of the inclusion of analytic rings in pre-analytic rings), we can form an analytic ring, say $\mathbb{Z}[\hat{q}^{\pm 1}]_{\text{gas}}$. Sending q to $\frac{1}{2}$ then gives a map of analytic rings $\mathbb{Z}[\hat{q}^{\pm 1}]_{\text{gas}} \to \mathbb{R}_{\text{gas}}$, while sending q to p gives a map $\mathbb{Z}[\hat{q}^{\pm 1}]_{\text{gas}} \to \mathbb{Q}_{p,\text{gas}}$ (which is different from $\mathbb{Q}_{p,\square}$ but it is good enough to do non-archimedean geometry).

Thus $\mathbb{Z}[\hat{q}^{\pm 1}]_{\text{gas}}$ is an analytic ring that specialises to both $\mathbb{Q}_{p,\text{gas}}$ and $\mathbb{R}_{\text{gas}}$: this object plays an important role in the description of the gaseous base stack.

5 Six-functor formalism for Analytic Stacks - by Adam Dauser

The first step is to talk about the 6-functor formalism. The second step is to talk about analytic stacks: here the term *analytic* means that these objects are modelled on analytic rings, while the term *stacks* can be thought as a generalisation of the notion of schemes.

5.1 Introduction to six-functor formalisms

The first six functor formalism was constructed by Grothendieck et al. [2] to handle (relative) étale cohomology (with compact supports). Let's briefly recall how it goes (with coefficients in $\Lambda = \mathbb{Z}/n\mathbb{Z}$, $\Lambda = \mathbb{Z}_\ell$ or $\Lambda = \mathbb{Q}_\ell$, where n resp. ℓ is tacitly assumed to be a unit on all schemes):

1. To a scheme X, associate the étale site $X_{\text{ét}}$ consisting of étale maps $f : Y \to X$ with covers given by jointly surjective maps. Form the subcategory $\mathrm{D}(X_{\text{ét}})$ of the derived category[1] of sheaves of Λ-modules on X whose cohomology sheaves are constructible;

2. $\otimes_\Lambda^{\mathrm{L}}$ equips $\mathrm{D}(X_{\text{ét}})$ with the structure of a symmetric monoidal category, and we can form $\mathrm{R}\underline{\mathrm{Hom}}_\Lambda$ as its right-adjoint;

3. pulling back sheaves along a map $f : X \to Y$ yields $f^* : \mathrm{D}(Y_{\text{ét}}) \to \mathrm{D}(X_{\text{ét}})$ with right adjoint, the *relative étale cohomology "push forward" functor* $\mathrm{R}f_*$;

4. if $j : U \to X$ is an open immersion, j^* has a left adjoint, the *extension by zero functor* $j_!$. Factoring a separated map of finite type $f : X \to Y$ as

$$X \overset{j}{\hookrightarrow} \overline{Y} \xrightarrow{\overline{f}} Y$$

   an open immersion followed by a proper morphism, we define the *relative étale cohomology with compact support "exceptional push forward" functor* $\mathrm{R}f_! \coloneqq \mathrm{R}f_* j_!$, which admits a right adjoint $\mathrm{R}f^!$. These are the six functors.

5. We have the projection formula: given a separated map of finite type $f : X \to Y$, there is a "canonical" isomorphism $\mathrm{R}f_!(A \otimes_\Lambda^{\mathrm{L}} f^* B) \simeq (\mathrm{R}f_! A) \otimes_\Lambda^{\mathrm{L}} B$ functorial in A and B;

6. and proper base change: Given a fibre square

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"] & X \arrow[d, "g"] \\
Y' \arrow[r, "f"] & Y,
\end{tikzcd}$$

---

[1] Starting here, we will treat all derived categories as ∞-categories.

there is a "canonical" isomorphism $g^*\mathrm{R}f_!(A) \simeq \mathrm{R}f'_!g'^*(A)$ functorial in A.

Remark 5.1 (First Caveat)**.** The meaning of the word *canonical* here is a bit subtle. These isomorphisms are additional data and they have to satisfy some compatibility conditions which are quite involved. This problem is worse yet for analytic stacks. Yifeng Liu and Weizhe Zheng [4] give a precise formulation of all this data, which has been nicely repackaged by Lucas Mann [5].

As notation like $\mathrm{R}f^!$ is a bit of a red herring anyway, we will drop R's and L's from the notation.

5.2 Six-functors for quasi-coherent sheaves on affine schemes

Everything can (almost) be translated to quasi-coherent sheaves on affine schemes:

1. To any ring R associate $\mathrm{D}(\mathrm{Spec}(R)) := \mathrm{D}(\mathrm{Mod}_R)$, i.e. the derived category of quasicoherent sheaves on $X = \mathrm{Spec}(R)$ (here Mod_R can denote either discrete R-modules or condensed ones);
2. we have $\otimes_R^{\mathrm{L}}$ and $\mathrm{R}\underline{\mathrm{Hom}}_R$;
3. for $f : \mathrm{Spec}(R) \to \mathrm{Spec}(S)$, we have $f^* := - \otimes_S^{\mathrm{L}} R : \mathrm{D}(\mathrm{Spec}(S)) \to \mathrm{D}(\mathrm{Spec}(R))$ with right adjoint $(-)_S =: f_*$ (which is the restriction to S);
4. we set $f_! := f_*$, which admits a right adjoint $f^! := \mathrm{RHom}_R(S, -)$;[2]
5. there is a canonical isomorphism $f_!(A \otimes f^*B) \simeq (f_!A) \otimes B$ functorial in A and B;
6. given a fibre square

$$\begin{tikzcd}
\mathrm{Spec}(S' \otimes_S R) \arrow[r, "f'"] \arrow[d, "g'"] & \mathrm{Spec}(R) \arrow[d, "g"] \\
\mathrm{Spec}(S') \arrow[r, "f"] & \mathrm{Spec}(S),
\end{tikzcd}$$

such that g is *flat*, there is a canonical isomorphism $g^*f_!(A) \simeq f'_!g'^*(A)$ functorial in $A \in \mathrm{D}(Y')$.

Remark 5.2 (Second Caveat)**.** One can eliminate the condition that g is flat by working with animated rings[3] instead—this means we would be taking $\mathrm{Spec}(S' \otimes_S^{\mathrm{L}} R)$ instead. Then we obtain a quasi-coherent six functor formalism on quasi-coherent sheaves for affine schemes.

In the following we will also secretly work with animated analytic rings/stacks.

It would be nice to glue these functors together to all schemes, to obtain a six-functor formalism. This is tricky: we do this by passing to stacks.

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[2] Not to be confused with the functor from Grothendieck duality.

[3] These can e.g. modelled via simplicial rings

5.3 Passing to stacks

"Stacks" might sound a bit scary at first, but we will treat them analogously to the natural continuation of the tower of generalisations

$$\text{affine schemes} \to \text{separated qc schemes} \to \text{schemes}$$

given by the following:

- separated (qc) schemes X are exactly those sheaves in the Zariski topology that admit a (finite) Zariski cover $\widetilde{X} \to X$ by an affine scheme and such that the map $\Delta : X \to X \times X$ is affine (i.e. for every morphism $\mathrm{Spec}(S) \to X \times X$, the fiber product $\mathrm{Spec}(S) \times_{X \times X} X$ along Δ is an affine scheme).
- Similarly, schemes are those Zariski sheaves admitting a Zariski cover $\coprod_{i \in I} \widetilde{X}_i \to X$ where each $\widetilde{X}_i$ is a separated qc scheme and such that $\Delta : X \to X \times X$ is quasi-compact and separated (i.e. for every morphism $Y \to X \times X$ with Y qc separated, the fiber product $Y \times_{X \times X} X$ along Δ is qc separated).

Exchanging the Zariski for the étale topology, this would give us

$$\begin{aligned} \text{affine schemes} &\to \text{qc algebraic spaces with affine diagonal} \\ &\to \text{algebraic spaces} \to \text{Deligne-Mumford stacks} \to \cdots \end{aligned}$$

(the list now proceeds infinitely to the right if we allow sheaves of 2-groupoids, ..., anima instead).

The technically most important fact about these topologies is that $\mathrm{D}(-)$ satisfies descent for them. This fact allows us to extend the six-functor formalism for quasi-coherent sheaves from affine schemes to schemes. We will see later how.

For us, affine schemes are replaced by analytic rings (i.e. affine analytic stacks) and the Zariski (or étale) topology is replaced by the !-topology. This topology is roughly defined to be the finest manageable topology for which $\mathrm{D}(-)$ satisfies descent.

In the world of analytic stacks, there is just no relevant distinguished class of "analytic schemes", because the !-toplogy is a super fine topology (for usual schemes, it is way stronger than the fppf topology).

Before studying this topology, we start by building a six-functor formalism on quasi-coherent sheaves of affine analytic stacks.

5.4 Six functors on affine analytic stacks

Let AnRing be the category of analytic rings. We define the category of *affine analytic stacks* as

$$\mathrm{AffAnStack} \coloneqq \mathrm{AnRing}^{\mathrm{op}},$$

where the affine analytic stack corresponding to R is denoted by $\mathrm{AnSpec}(R)$.

The first three elements necessary for a six-functor formalism can be easily defined as follows:

1. To an analytic ring $R = (R^{\triangleright}, \mathrm{D}(R))$, we associate $\mathrm{D}(\mathrm{AnSpec}(R)) \coloneqq \mathrm{D}(R)$, which can be thought as the category of quasi-coherent sheaves on $\mathrm{AnSpec}(R)$.
2. By Proposition 4.4 we have a functor $- \otimes^{\mathrm{L}}_{R} -$ with right adjoint $\mathrm{R}\underline{\mathrm{Hom}}_{R}(-,-)$.
3. For $f : \mathrm{AnSpec}(R) \to \mathrm{AnSpec}(S)$ (corresponding to a morphism of analytic rings $S \to R$), by Proposition 4.4 we have a functor $f^* \coloneqq - \otimes^{\mathrm{L}}_{S} R : \mathrm{D}(S) \to \mathrm{D}(R)$ with right adjoint $f_* \coloneqq (-)_S : \mathrm{D}(R) \to \mathrm{D}(S)$.

The non-trivial part is now to define the functors $f^!$ and $f_!$ for a suitable class of maps.

Let $f : \mathrm{AnSpec}(R) \to \mathrm{AnSpec}(S)$ be a map of affine analytic stacks. Here we have $R \coloneqq (R^{\triangleright}, \mathrm{D}(R))$ and $S \coloneqq (S^{\triangleright}, \mathrm{D}(S))$. We observe that f can be factored into

$$\mathrm{AnSpec}(R) \overset{j}{\hookrightarrow} \mathrm{AnSpec}(R^{\triangleright}_{/S}) \xrightarrow{\overline{f}} \mathrm{AnSpec}(S), \tag{2}$$

where $R^{\triangleright}_{/S} = (R^{\triangleright}, \mathrm{D}(R^{\triangleright}_{/S}))$ is the *induced analytic ring structure* of Example 4.7. (Note that if $R^{\triangleright} \notin \mathrm{D}(S)$ this only defines a pre-analytic ring. In this case, one would need to complete $R^{\triangleright}$ with respect to this pre-analytic ring structure.)

Definition 5.3. Let $f : \mathrm{AnSpec}(R) \to \mathrm{AnSpec}(S)$ be a map of affine analytic stacks.

1. We say that f is *proper* if we have $f = \overline{f}$.
2. We say that f is an *open immersion* if we have $f = j$ and moreover j^* admits a fully faithful left adjoint $j_!$ satisfying the projection formula.
3. We say that f is !-able if in the factorisation (2) j is an open immersion.

Remark 5.4. Warning: By definition, any map of discrete rings with the trivial analytic ring structure is proper. In particular, these words do not match up with the nomenclature of algebraic geometry. This may seem strange but actually it is not. For example, this fact explains why in algebraic geometry we don't need the *properness* assumption for the qcqs proper base change.

We now have a six-functor formalism on affine analytic stacks:

1. if $X = \mathrm{AnSpec}(R)$, we set $\mathrm{D}(X) \coloneqq \mathrm{D}(R)$.
2. We have functors $- \otimes^{\mathrm{L}}_{R} -$ and $\mathrm{R}\underline{\mathrm{Hom}}_{R}(-,-)$.
3. If $f : \mathrm{AnSpec}(R) \to \mathrm{AnSpec}(S)$, we have $f^* \coloneqq - \otimes^{\mathrm{L}}_{S} R : \mathrm{D}(S) \to \mathrm{D}(R)$ with right adjoint $f_* : \mathrm{D}(R) \to \mathrm{D}(S)$ (the restriction functor).
4. If f is !-able and j and $\overline{f}$ are as in (2), we set $f_! \coloneqq \overline{f}_* \circ j_!$. This functor admits a right adjoint $f^!$. Explicitly we have $f^! = \mathrm{R}\,\mathrm{Hom}_R(S, -)$ for proper maps and $f^! = j^*$ for open immersions;

5. if f is !-able, there is a canonical isomorphism $f_!(A \otimes f^*B) \simeq (f_!A) \otimes B$ functorial in $A \in \mathrm{D}(R)$ and $B \in \mathrm{D}(S)$;

6. given a fibre square

$$\begin{tikzcd}
\mathrm{Spec}(S' \otimes^{\mathrm{L}}_S R) \arrow[r, "f'"] \arrow[d, "g'"] & \mathrm{Spec}(R) \arrow[d, "g"] \\
\mathrm{Spec}(S') \arrow[r, "f"] & \mathrm{Spec}(S)
\end{tikzcd}$$

where f is !-able, there is a canonical isomorphism $g^* f_!(A) \simeq f'_! g'^*(A)$ functorial in $A \in \mathrm{D}(S')$.

5.5 Passing to analytic stacks

As we anticipated, to glue analytic rings we use the !-topology, a super fine topology roughly defined to be the finest manageable topology for which $\mathrm{D}(-)$ satisfies descent. Here is a useful criterion to determined whether a proper map is a !-cover

Proposition 5.5. *Let $f : \mathrm{AnSpec}(B) \to \mathrm{AnSpec}(A)$ be a !-able map. Suppose that f is proper. Then f is a !-cover if and only if it is* descendable*, i.e. we have*

$$A \in \langle f_*M \,|\, M \in \mathrm{D}(B) \rangle_{\text{fin. limits, retracts, } \otimes}.$$

Remark 5.6. If $A \to B$ is a map between discrete rings, writing

$$A \to B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \dots,$$

we observe that this descendability condition is *almost* satisfied for faithfully flat maps.

Example 5.7. Countably presented faithfully flat maps between discrete rings are !-covers.

We are now ready to glue affine analytic stacks using the !-topology, to define analytic stacks.

Definition 5.8. An analytic stack is a sheaf[4] of anima AnRing → Ani, where AnRing is endowed with the !-topology.

We denote AnStack the category of analytic stacks: our goal is to extend the 6-functor formalism from $\mathrm{AnRing}^{\mathrm{op}}$ to AnStack.

As an enlightening exercise, let us first see how to extend the six-functor formalism of Section 5.2 from affine schemes to quasi-compact separated schemes using the Zariski topology, following the discussion started in Section 5.3.

---

[4] Often a slightly technical notion between sheaf and hypersheaf is assumed instead.

1. If X is a quasicompact separated scheme and $U \coloneqq \bigsqcup_i U_i \to X$ is a Zariski cover where U and U_i are all affine, define $\mathrm{D}(X) \coloneqq \varprojlim_i \mathrm{D}(U_i)$;
2. $\mathrm{D}(X)$ automatically becomes equipped with $- \otimes^{\mathrm{L}}_{\mathcal{O}_X} -$ and $\mathrm{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}(-,-)$;
3. for each map $f : X \to Y$ between quasi-compact separated schemes, take a finite open affine cover $\{U_i\}$ of Y and a compatible one $\{V_{ij}\}$ on X. Then we can glue the pullback maps $\mathrm{D}(U_i) \to \mathrm{D}(V_{ij})$ to obtain $f^* : \mathrm{D}(Y) \to \mathrm{D}(X)$. This is independent of the covers by Zariski descent. We obtain f_* as its right adjoint;
4. Defining $f_!$ is a bit more subtle now.
    - If $f : X \to Y$ is an affine map between quasi-compact separated schemes, the definition of $f_!$ is forced by the fact that we want proper-base change to hold. Indeed, if we have $U \hookrightarrow Y$ with U affine, then $f^{-1}(U)$ is affine as well and we consider the fibre square

      $$\begin{tikzcd} f^{-1}(U) \arrow[r, "f'"] \arrow[d, "i'"] & U \arrow[d, "i"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

      Since proper base change must hold, we get

      $$i^* f_! = f'_! i'^* = f'_* i'^*,$$

      where the last equality comes from the fact that we defined $f'_! \coloneqq f'_*$ for all morphisms of affine schemes. Thus if we take an open cover $\{U_i\}$ of Y, we obtain $f_i : f^{-1}(U_i) \to U_i$ and we can define $f_! : \mathrm{D}(X) \to \mathrm{D}(Y)$ via descent from $f_{i,!} \coloneqq f_{i,*}$. This admits a right adjoint $f^!$ and automatically satisfies proper base change and projection formula (see [5, Proposition A.5.12]).
    - If $f : X \to Y$ is any map between quasi-compact separated schemes, there exists a open affine cover $j_i : V_i \to X$ such that the morphisms $g : \bigsqcup_i V_i \to X$ and $f \circ g : \bigsqcup_i V_i \to Y$ are both affine. [5] We set n-fold self intersections

      $$g_n : \bigsqcup_{(i_1,\dots,i_n)\in I^n} V_{i_1} \cap \dots \cap V_{i_n} \to Y.$$

      One can show that the canonical morphism $\varinjlim_{n\in\Delta} g_{n,!} g_n^! \to \mathrm{id}$ is an equivalence. This leads us to define by descent

      $$f_! \coloneqq \varinjlim_{n\in\Delta} (f \circ g_n)_! g_n^!. \tag{3}$$

      Again, this admits a right adjoint $f^!$.

---

[5] Indeed, find an open affine cover $U_i \to Y$ such that $\bigsqcup_i U_i \to Y$ is affine, pulling back yields that $\bigsqcup_i f^{-1}(U_i) \to X$ is affine. Now we may find finite open affine covers $V_{ij} \to f^{-1}(U_i)$ of the quasi-compact spaces $f^{-1}(U_i)$.

5. Proper base change and projection formula are automatically satisfied, see [5, Proposition A.5.12].

We now run this program for analytic stacks, obtaining the following:

1. For any analytic stack $\mathcal{X}$, we set

$$\mathrm{D}(\mathcal{X}) \coloneqq \lim_{\substack{(X\to\mathcal{X})\\ X \text{ affine}}} \mathrm{D}(X),$$

   (a quasi-coherent sheaf on $\mathcal{X}$ can be pulled back to X);

2. $-\otimes-$ and $\underline{\mathrm{Hom}}(-,-)$ are automatic;

3. For a map $f : \mathcal{X} \to \mathcal{Y}$ between analytic stacks, we define $f^* : \mathrm{D}(\mathcal{Y}) \to \mathrm{D}(\mathcal{X})$ and its right adjoint f_* by descent;

4. For a mysterious class of maps of analytic stacks that are called !-able (this class contains !-able maps between affine analytic stacks as in Definition 5.3), we define $f_!$ as in the previous example. More precisely:

   - we first define $f_!$ for affine maps between analytic stacks: this can be done by descent from the definition of $f_!$ for !-able maps between analytic rings;
   - then we define $f_!$ inductively as in (3).

   This map has a right adjoint that we call $f^!$.

5. These maps satisfy proper base change and projection formula.

Remark 5.9. Note that if we have an $g : U \to Y$ open cover between schemes which is *not* quasi-compact, the functor $j^* : QCoh(Y) \to QCoh(U)$ between the categories of discrete (i.e. non-condensed) quasi-coherent sheaves does not admit a left-adjoint $j_!$ as it does not preserve arbitrary products. This problem is fixed when passing to condensed quasi-coherent sheaves and we can again utilize our trusty formula

$$f_! \coloneqq \varprojlim_{n\in\Delta} (f \circ g_n)_! g_n^!.$$

Note that this functor does not preserve discrete quasi-coherent sheaves in general.

5.6 More from the discussion session

During the discussion session, some topics from this talk have been addressed in more detail. We present the most relevant comments in this section.

5.6.1 !-covers, definition and properties

Definition 5.10. A map $f : X \to Y$ between analytic stacks is a !-cover if

- f is !-able;
- f satisfies the *universal* $*$-descent, i.e. if we set $X^{/^n_Y} := X \times_Y \cdots \times_Y X$ we have

  $$\mathrm{D}(Y) \xrightarrow{\sim} \varprojlim_{n \in \Delta} \mathrm{D}(X^{/^n_Y}),$$

  where the morphisms in the limit diagram are pullbacks π_i^*. Moreover, if we have a fibre square

  $$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y, \end{tikzcd}$$

  the same property is satisfied by $f' : X' \to Y'$ (this is universality).
- f satisfies the universal !-descent, i.e. we have

  $$\mathrm{D}(Y) \xrightarrow{\sim} \varprojlim_{n \in \Delta}{}^{!}\ \mathrm{D}(X^{/^n_Y}),$$

  where $\lim^!$ denotes the fact that the morphisms in the limit diagram are not simple pullbacks π_i^* but exceptional pullbacks $\pi_i^!$. Moreover, the same holds for all pullbacks $f' : X' \to Y'$ as before (universality).

Remark 5.11. A !-able map is a !-cover if and only if it satisfies !-descent. Indeed, in the context of analytic stacks, if f satisfies the !-descent, then it satisfies $*$-descent and universal !-descent (hence universal $*$-descent is automatic as well).

Here is a condition to check that a map is a !-cover.

Proposition 5.12. *Let $f : X \to Y$ be a !-able map of analytic stacks.*

1. *If f is a !-cover, then we have $1_{\mathrm{D}(Y)} \in \langle \mathrm{im}(f_!) \rangle$.*
2. *The converse is true if, e.g.*
   - *f is proper;*
   - *$f^!$ is "nice enough" (e.g. if f is cohomologically smooth, i.e. $f^! \simeq f^* \otimes \mathrm{id}$, or if f is given by a finite family of open immersions).*

To convince ourselves why 1 holds, we should observe that !-descent implies that setting $f_n : X^{/^n_Y} \to Y$ we have

$$1_{\mathrm{D}(Y)} = \varinjlim_{n \in \Delta} f_{n,!} f_n^! 1_{\mathrm{D}(Y)},$$

(i.e. id is the filtered colimit of partial truncations of $f_!$).

5.6.2 Open immersions and complementary idempotent algebras

Open immersions $j : \mathrm{AnSpec}(R) \to \mathrm{AnSpec}(S)$ are closely related to *idempotent algebras* in $\mathrm{D}(S)$, as it is justified by the following

Remark 5.13. If $j : U \hookrightarrow X$ is an open immersion of schemes, then $j_!\Lambda \in \mathrm{D}(X_{\text{ét}})$ is an idempotent co-algebra: There is a co-unit $\epsilon : j_!\Lambda \to \Lambda$ and co-multiplication $c : j_!\Lambda \overset{\simeq}{\to} j_!\Lambda \otimes j_!\Lambda$. $\mathrm{D}(U_{\text{ét}})$ is exactly the category of co-modules over this co-algebra i.e. sheaves such that $\mathcal{F} \overset{\simeq}{\to} \mathcal{F} \otimes j_!\Lambda$. If the complement[6] is $i : Z \to X$, $i_*\Lambda$ is similarly an idempotent algebra and we can recover $j_!\Lambda$ as the fibre

$$j_!\Lambda \to \Lambda \to i_*\Lambda.$$

The same is true for analytic rings: Open immersions of analytic stacks come from "complementary" idempotent algebras $A \in \mathrm{D}(S)$. Indeed, we have the following

Proposition 5.14. *Let* $j : \mathrm{AnSpec}(R) \to \mathrm{AnSpec}(S)$ *be a morphism of affine analytic stacks. Then* j *is an open immersion if and only if there exists an idempotent algebra* $A \in \mathrm{D}(S)$ *such that* $\mathrm{Ker}(j^*) = \mathrm{Mod}_A(\mathrm{D}(S))$.

Remark 5.15. In general, it is not true that *all* idempotent algebras A in $\mathrm{D}(S)$ are complementary to open immersions of affine analytic stacks. To ensure this, one needs that the functor $R\mathrm{Hom}_{D(S)}(A, -)[1]$ preserves colimits and $D_{\geq 0}(S)$. Otherwise, the derived category $\mathrm{D}(S)/\mathrm{Mod}_A(S)$ would *not* be the derived category of an analytic ring. This situation corresponds to an open immersion $j : X \to \mathrm{AnSpec}(S)$ of analytic stacks where the source is not affine.

5.6.3 Example of !-able map

Example 5.16 (Example of !-able map)**.** Let us consider $f : \mathrm{AnSpec}(\mathbb{Z}[T]_\square) \to \mathrm{AnSpec}(\mathbb{Z}_\square)$. The decomposition (2) here is

$$\mathrm{AnSpec}(\mathbb{Z}[T]_\square) \xrightarrow{j} \mathrm{AnSpec}(\mathbb{Z}[T]_{/\mathbb{Z}_\square}) \xrightarrow{\overline{f}} \mathrm{AnSpec}(\mathbb{Z}_\square).$$

In order to conclude that f is !-able, we just need to show that j is an open immersion. This can be deduced by Proposition 5.14. Indeed, at the end of Section 4.2 we observed that we have

$$j^* = - \otimes_{\mathbb{Z}[T]_{/\mathbb{Z}_\square}} \mathbb{Z}[T]_\square = R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathrm{fib}\big(\mathbb{Z}[T] \to \mathbb{Z}((T^{-1}))\big), -)$$

so that $A = \mathbb{Z}((T^{-1})) \in \mathrm{D}(\mathbb{Z}[T]_{/\mathbb{Z}_\square})$ is an idempotent algebra and $\mathrm{Ker}(j^*) = \mathrm{Mod}_A(\mathrm{D}(\mathbb{Z}[T]_{/\mathbb{Z}_\square}))$. Hence j is an open immersion which has, as a complement idempotent algebra, $\mathbb{Z}((T^{-1}))$, the formal open disk at ∞.

More generally, if R and S are finite-type $\mathbb{Z}$-algebras together with a morphism $S \to R$, then the morphism $\mathrm{AnSpec}(R_\square) \to \mathrm{AnSpec}(S_\square)$ is !-able.

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[6] With any scheme structure e.g. the reduced one.

Example 5.17 (Example of closed immersion)**.** Let R be a discrete ring and $f \in R$ which is not a zero-divisor. Then the morphism

$$\mathrm{AnSpec}(R[1/f]_{\mathrm{triv}}) \to \mathrm{AnSpec}(R_{\mathrm{triv}})$$

is a *closed* immersion (even if it is a Zariski open!). Indeed, we have

$$\mathrm{D}(R[1/f]_{\mathrm{triv}}) = \mathrm{Mod}_{R[1/f]}(\mathrm{D}(R_{\mathrm{triv}})),$$

where $R[1/f]$ is an idempotent algebra in $\mathrm{D}(R_{\mathrm{triv}})$. The open complement is the ind-completion at $f = 0$, which is *not* affine (it is an example of the situation described in Remark 5.15).

6 An introduction to Higher dimensional Arakelov theory- by José Burgos Gil

Introduction

Main objects in Arakelov geometry are varieties defined over number fields. In this talk, we will study the special case of a smooth projective variety defined over $\mathbb{Q}$.

Arakelov Gemetry studies several quantities associated to X :

- A **model** $\mathcal{X}$ of X defined over $\mathbb{Z}$. Informally speaking, as X is projective we just have to clear the denominators in the equations defining X in $\mathbb{P}^n_{\mathbb{Q}}$. To be precise, as we have the open immersion $\mathrm{Spec}(\mathbb{Q}) \hookrightarrow \mathrm{Spec}(\mathbb{Z})$, we can take the Zariski closure of X seen as a subscheme of $\mathbb{P}^n_{\mathbb{Z}}$.
- The **complex points** of X, namely $X(\mathbb{C})$.
- Points at infinity in X correspond to archimedian norm. We have an **involution** F_∞ of $X(\mathbb{C})$ acting on them, it defines a real structure on X.

The data of $(\mathcal{X}, X(\mathbb{C}), F_\infty)$ should behave like a complete variety over a field. So all classical theorems in algebraic geometry such as Bezout's theorem should have an analog in Arakelov geometry. To construct this analog, we have to rely classical algebraic geometry (on $\mathcal{X}$), and complex analysis (on $X(\mathbb{C})$). In these notes we will be interested in the Grothendieck-Riemann-Roch formula

6.1 Geometric case

Let X be a smooth projective variety over $\mathbb{C}$. We first need to define the main ingredients of the theorem : vector bundles and algebraic cycles.

6.1.1 Grothendieck ring

Definition 6.1. Let Bun_X be the set of vector bundles on X. In the free abelian group $\mathbb{Z} < Bun_X >$ we define the equivalence relation $[E] \sim [E'] + [E'']$ for any short exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

The quotient $K_0(X) := {}^{\mathbb{Z}[Bun_X]}/_{\sim}$ is called the Grothendieck group.

Obviously we have $[E \oplus E'] = [E] + [E']$.

Proposition 6.2. *The tensor product $\otimes$ induces a ring structure on $K_0(X)$ via $[E] \cdot [E'] = [E \otimes E']$. It is even a λ-ring, which means that we have (anti)-symmetric operators, and other Young operators from their definition in the category of vector bundles.*

6.1.2 Algebraic cycles

Definition 6.3. $Z^p(X)$ is the free abelian group generated by closed irreducible subvarieties $Z \subset X$ of codimension p. $Rat^p(X)$ is the subgroup generated by classes of the form $div(f)$ for f a rational function on some (irreducible normal) subvariety $W \subset X$ of codimension $p-1$. Finally we define the p Chow group by

$$CH^p(X) := {}^{Z^p(X)}/_{Rat^p(X)}$$

and the total Chow group by $CH^*(X) := \bigoplus_{p \geq 0} CH^p(X)$.

As in the K-theory world, we have a (non trivial) ring structure.

Theorem 6.4. *As X is smooth projective, we can define a ring structure on $CH^*(X)$ via intersection theory.*

6.1.3 Operators

For $f : X \to Y$ of relative dimension e we want to define pushforward and pullback. In each theory one operator is easy to describe and the other is more subtle.

Definition 6.5. We define the additive operator $f_* : CH^p(X) \to CH^{p-e}(Y)$ by

$$f_*[Z] := \begin{cases} [K(Z) : K(f(Z))]f(Z) \text{ if } dim(Z) = dim(f(Z)), \\ 0 \text{ otherwise.} \end{cases}$$

The operator f^* is more difficult to define.

Proposition 6.6. *There exist a pullback map f^* compatible with the graduation of $CH^*(X)$ and the ring structure. When f is flat it is just $f^*[Z] := [f^{-1}(Z)]$.*

Once these operators are defined, we have all the usual formalism such as the adjunction formula.

For the Grothendieck group, we can use the following definition.

Definition 6.7. the pullback operator is defined by $f^*[E] = [f^*E]$. It is a ring morphism.
For the pushforward map, we define

$$f_*[E] := \sum_{i \geq 0} (-1)^i [R^i f_* E].$$

A priori $R^i f_* E$ is just a coherent sheaf, but we use the smoothness assumption on X to replace $R^i f_* E$ by a finite resolution with vector bundles, so the definition makes sense.

Again we have the usual formalism.

Remark 6.8. As f_* is easy to define in the theory of algebraic cycles and f^* is hard, we can say that $CH^*(X)$ is like an homological theory. On the other hand $K_0(X)$ is similar to a cohomological theory for the opposite reason.

We now have two theories on X and we want to compare them. For this we have to define Chern classes.

6.1.4 Characteristic classes

For $[E] \in K_0(X)$ we can define the **Chern class** $c_i(E) \in CH^i(X)$. Up to some flat base change, the definition of this class is essentially contained in the following example.

Example 6.9. For L a line bundle, $c_1(L) = [div(s)]$ for any meromorphic section s ; and higher Chern classes vanish.

The class of a sum and a product can be computed, but it is not the sum/product of the classes. To have a nice morphism, we have to define a new quantity which is the **Chern character** $ch(E)$. Again the spirit of the definition is contained in the following example.

Example 6.10. For a line bundle L, $ch(L) = exp(c_1(L)) \in CH^*(X)_{\mathbb{Q}}$.

So the map ch is defined only after tensoring $CH^*(X)$ with $\mathbb{Q}$.

Proposition 6.11. *This description of the Chern character gives a morphism of ring*

$$ch_{\mathbb{Q}} : K_0(X) \to CH^*(X)_{\mathbb{Q}},$$

and for any map $f : X \to Y$, we have $ch(f^*E) = f^*(ch(E))$.

Theorem 6.12. *After tensoring the initial ring with $\mathbb{Q}$,*

$$ch_{\mathbb{Q}} : K_0(X)_{\mathbb{Q}} \to CH^*(X)_{\mathbb{Q}},$$

is an isomorphism.

Question. *Do we have $ch(f_*E) = f_*(ch(E))$ for any morphism f ?*

The answer is no ! We have to modify the morphism.

In fact we must define a **Todd Class** $Td(E) \in CH^*(X)_{\mathbb{Q}}$ such that : $Td(E \oplus E') = Td(E) \cdot Td(E')$. We thus have the following special case.

Example 6.13. For a line bundle L,

$$Td(L) = \frac{c_1(L)}{1 - exp(-c_1(L))} = \sum_{i \geq 0} (-1)^{i+1} \frac{B_i}{2i!} c_1(L)^{2i} \in CH^*(X)_{\mathbb{Q}}.$$

We now have the modified diagram :

$$\begin{tikzcd}
K_0(X) \arrow[r, "f_*"] \arrow[d, "ch(\cdot)Td(T_X)"'] & K_0(Y) \arrow[d, "ch(\cdot)Td(T_Y)"] \\
CH^*(X) \arrow[r, "f_*"'] & CH^*(Y)
\end{tikzcd}$$

where T_X is the (algebraic) tangent bundle on X.

Theorem 6.14 (Grothendieck-Riemann-Roch)**.** *This diagram commute (without assumption on f because we assume that X and Y are smooth).*

When Y is a point we recover the usual Riemann-Roch theorem, and when f is a finite morphism, we recover the Riemann-Hurwitz theorem (applied to $\mathcal{O}_X$).

6.2 Arakelov case

We want to generalize the previous section in the context of Arakelov geometry. The main idea is to put an hat everywhere ! We first have to define all the objects : on $\mathcal{X}$ and on $X(\mathbb{C})$. This new construction must take into account the hermitian metric on holomorphic vector bundle.

6.2.1 Vector bundles

Let E be a vector bundle over $X/\mathbb{Q}$. On $\mathcal{X}$ we just have to take $\mathcal{E}$ as a model of E over $\mathcal{X}$. (we need to shrink the model of X to have the existence of the model of E).

Question. *What should we do on $X(\mathbb{C})$? In particular $E(\mathbb{C})$ is equipped with an hermitian metric $\|\cdot\|$, how can we take into account this additional data ?*

Idea. *$\hat{K_0}(\mathcal{X})$ should be the group of pairs $(\mathcal{E}, \|\cdot\|)$ where $\mathcal{E}$ is a vector bundle on $\mathcal{X}$, equipped with a norm on $E(\mathbb{C})$, and quotiented by short exact sequences.*

In fact the definition will be more complicated. We begin by recalling standard facts in complex geometry.

Definition 6.15. Let X be a complex variety of dimension d. The data of a complex manifold gives a complex structure on the tangent space. Which means that we have an operator

$$J : T_X \to T_X$$

on the (differentiable)-vector bundle T_X, such that $J^2 = -Id$. So we can decompose $T_X \otimes \mathbb{C}$ in two subbundles corresponding to the eigenvalues i and $-i$,

$$T_X \otimes \mathbb{C} = T_X^{1,0} \oplus T_X^{0,1}.$$

We can now define the sheaf of differentials forms

$$\mathcal{A}_X^{p,q} := \bigwedge^p T_X^{1,0} \otimes \bigwedge^q T_X^{0,1}$$

Example 6.16. Locally a (p,q)-form can be written as

$$\omega = \sum_{|I|=p,|J|=q} f_{I,J} dz_I \wedge \overline{dz_J},$$

where $f_{I,J}$ is an analytic map.

Notation. *We note* $\mathcal{A}_X^m := \bigoplus_{p+q=m} \mathcal{A}_X^{p,q}$*, and* $\mathcal{A}_X^* := \bigoplus_m \mathcal{A}_X^m$.

We have an operator $d^n : \mathcal{A}_X^n \to \mathcal{A}_X^{n+1}$ with a $(1,0)$-part ∂ and a $(0,1)$-part $\overline{\partial}$. It is a differential map. The following computations are well-known.

Proposition 6.17. *We have the following identifications*

$$H^*(\mathcal{A}_X^*, d) = H_{dR}^*(X, \mathbb{C}),$$

and

$$H^*(\mathcal{A}_X^{p,*}, \overline{\partial}) = H^*(X, \Omega_X^p).$$

Under the $\partial\overline{\partial}$- lemma (for instance if X is projective) we recover a part of the Hodge decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^q(X, \Omega_X^p).$$

6.2.2 Chern classes

If E is a vector bundle with an hermitian metric $\|\cdot\|$, $\xi_1, \cdots, \xi_k$ a local frame, the hessian matrix is $H = (<\xi_i, \xi_j>)_{i,j}$.

Definition 6.18. The curvature is the following matrix of $(1,1)$-form

$$K = \frac{i}{2\pi} \overline{\partial}(H^{-1}\partial(H)).$$

If we change the local frame by a matrix A, the new curvature will be related to the old one by the following formula :

$$K' = A^{-1}KA.$$

So for any polynom of matrices invariant under this action of conjugaison, we can defined $P(E, \|\cdot\|)$ and this quantity won't depend on the choice of a (local) frame. We apply this technic in the next definition.

Definition 6.19. $c_i(E, \|\cdot\|) = tr_i(K)$ is a closed (i,i)-differential form, where $tr_i(M)$ is the sum of all i-power of eigenvalues in M (with multiplicity).

Proposition 6.20. *With the previous notation, the cohomological class of* $c_i(E, \|\cdot\|)$ *(which is in* $H^{i,i}(X, \mathbb{C})$*) does not depend on the choice of the metric. We denote this class by* $c_i(E)$.

When X is projective, this definition of $c_i(E)$ gives the same result as in the previous section (once we have applied the realization map $CH^i(X) \to H^{i,i}(X)$). Hence it is a nice analytic definition of the Chern classes. Furthermore $c_i(E, \|\cdot\|) \in \mathcal{A}_X^{i,i}$ is a refinement who keep track of the information on the metric.

Example 6.21. For $\overline{L} = (L, \|\cdot\||)$ a line bundle with a frame s, we find that the curvature is

$$K = \frac{i}{2\pi}\partial\overline{\partial}(-log(\|s\|^2))$$

Remark 6.22. In fact the curvature is more naturally associated to a connection. Here we have used the Chern connection associated to the hermitian bundle $(E, \|\cdot\|)$. In the context of differential geometry, the same strategy with Levi-Civita connection would have recovered the usual theory of curvature.

We can thus define an important space.

Definition 6.23.

$$\tilde{\mathcal{A}}_X^* := \oplus_{p\geq 0}\, \mathcal{A}_X^{p,p}/Im(\partial)+Im(\overline{\partial})$$

is a ring containing the Chern classes $c_i(E, \|\cdot\|)$.

6.2.3 Arithmetic Grothendieck group

We have defined the Chern classes for the pair $(E, \|\cdot\|)$. But unlike the classical case, the relation to Chern character is more subtle.

For L a line bundle and $\omega, \omega_0 \in c_1(L)$, $\omega - \omega_0$ is 0 in de Rham cohomology, so by $\partial\overline{\partial}$-lemma this difference is represented by $\frac{i}{2\pi}\partial\overline{\partial}f$. But for line bundles we can recover the metric from the data of ω ; so in that case, choosing a representant of the Chern class changes the metric by e^{-f}. So the choice of the metric is quite canonic, and it is essentially the same to consider $c_1(L)$ or $c_1(L, \|\cdot\|))$. However it is not the case for vector bundles of higher rank.

In an exact sequence ξ

$$\begin{tikzcd}
0 \arrow[r] & (E', \|\cdot\|') \arrow[r] & (E, \|\cdot\|) \arrow[r] & (E'', \|\cdot\|'') \arrow[r] & 0
\end{tikzcd}$$

we want to add an error term $\frac{i}{2\pi}\partial\overline{\partial}ch(\xi)$ to the naive equation

$$ch(E) = ch(E') + ch(E'')$$

coming from the data of the metric.

Proposition 6.24. *There is a class $ch(\xi) \in \tilde{\mathcal{A}}_X^*$ associated to a short exact sequence ξ such that*

$$ch(E) = ch(E') + ch(E'') + ch(\xi)$$

The class $ch(\xi)$ is called the **Bott-Chern character** of the exact sequence, and is functorial under pullback operator. It is unique up to the image of the operators ∂ and $\overline{\partial}$. We can now give the exact definition of the Grothendieck group

Definition 6.25. The arithmetic Grothendieck group is defined by

$$\hat{K}_0(\mathcal{X}) := {}^{(\mathbb{Z}<[E,\|\cdot\|]>\oplus\tilde{\mathcal{A}}^*_X)}/_{\sim}$$

where the relation $\sim$ identifies for any exact sequence ξ

$$[\mathcal{E}', \|\cdot\|'] + [\mathcal{E}'', \|\cdot\|''] = [\mathcal{E}', \|\cdot\|] + ch(\xi).$$

We know a very special case where the error term is vanishing.

Proposition 6.26. *If the hermitian short exact sequence is orthogonaly splitted, then* $ch(\xi) = 0$.

Finally we have a morphism $\hat{ch} : \hat{K}_0(X) \to \tilde{\mathcal{A}}^*$ sending $[E, \|\cdot\|]$ to its usual Chern class, and an element $\eta \in \tilde{\mathcal{A}}^*_X$ to itself.

In particular if $\omega \in ch(E)$, then there is $(E, \|\cdot\|, \eta)$ such that $\hat{ch}(E, \|\cdot\|, \eta) = \omega$.

6.2.4 Complex of currents

Now we want to study algebraic cycles on X, but mixed with the data of differential forms. So we want to construct a complex of currents which contain both these informations.

Definition 6.27. For X smooth projective of dim d on $\mathbb{C}$, we define $D_{p,q}(U)$ as the topological dual of $\mathcal{A}^{p,q}_c(U)$ (where the topology is the natural one taking into account the decreasing of partial derivatives).

Now let $D^{p,q}(U) = D_{d-p,d-q}(U)$. It form a sheaf $\mathcal{D}^{p,q}$ (analog to the situation with distribution).

The restriction to compactly support sections is essentially to obtain a sheaf. We can now see how to inject the data of usual differential forms in this complex.

Definition 6.28. We construct a map $\mathcal{A}^{p,q} \to \mathcal{D}^{p,q}$ such that ω is sent to $[\omega]$, where

$$[\omega] : \eta \to \int_X \omega \wedge \eta.$$

Then we can see how to add the information on algebraic cycles.

Definition 6.29. For $Y \in Z^p(X)$ we can associate $\delta Y \in D^{p,p}(U)$ such that

$$\delta Y : \eta \to \int_Y \eta.$$

In both definition, the integral is well defined because η is compactly supported.

Notation.
$$\mathcal{D}^m := \bigoplus_{m=p+q} \mathcal{D}^{p,q}$$

and

$$\tilde{\mathcal{D}}^* := \oplus_p \, \mathcal{D}^{p,p} \big/ Im(\partial)+Im(\overline{\partial})$$

To form a complex, we use the following definition.

Definition 6.30. We define a differential $d : \mathcal{D}^n \to \mathcal{D}^{n+1}$ by

$$dT(\eta) = (-1)^{n+1} T(d\eta).$$

It can be decomposed as $d = \partial + \overline{\partial}$.

As in the usual Hodge theory, this complex can be use to compute some useful cohomology groups.

Proposition 6.31. *We have the following equalities*

- $H^n(\mathcal{D}^*, d) = H^n_{dR}(X, \mathbb{C})$.
- $H^q(\mathcal{D}^{p,*}, \overline{\partial}) = H^q(X, \Omega^p)$.

Example 6.32. Let compute the differential of $\omega := [\frac{1}{2i\pi}\frac{dz}{z}]$ over $X = \mathbb{P}^1$. Due to our conventions, $d\omega$ applies to a function f and is value is

$$\frac{1}{2i\pi}\int_{\mathbb{P}^1} \frac{dz}{z} \wedge df = -\frac{1}{2i\pi}\int_{\mathbb{P}^1 - \{0,\infty\}} d\left(f\frac{dz}{z}\right)$$

By Stokes formula the quantity is equal to

$$\lim_{\epsilon \to 0^+} \frac{1}{2i\pi}\int_{\partial B(0,\epsilon)} f dz/z - \frac{1}{2i\pi}\int_{\partial B(0,1/\epsilon)} f dz/z$$

In the local chart $z = \epsilon e^{i\theta}$ we have $dz/z = id\theta$. So the previous computation gives $f(0) - f(\infty)$, hence $d\omega$ is the Dirac operator $\delta_{div(z)}$ (multiplicities in the divisor give the coefficients in front of each Dirac term).

More generally we have the following result.

Proposition 6.33. *If f is a rational function on X, $d[\frac{1}{2i\pi}\frac{df}{f}] = \delta_{div(f)}$, where δ is the Dirac operator.*

6.2.5 Green Current

Notation. $d^c := \frac{i}{4\pi}(\overline{\partial} - \partial)$. *We thus have* $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$.

The following equation is useful to compute the Chern class of a line bundle.

Proposition 6.34 (Poincaré-Lelong formula)**.** *For* L *a line bundle,* $\|\cdot\|$ *a smooth hermitian metric over* L *and* s *a rational section, we have*

$$dd^c(-log(\|s\|^2)) = -\delta_{div(s)} + c_1(L, \|\cdot\|).$$

Definition 6.35. Let $Z \in Z^p(X)$. A green current for Z is a current $g \in \tilde{\mathcal{D}}^{p-1,p-1}$ such that

$$\omega(g) := dd^c(g) + \delta_Z \in \tilde{\mathcal{A}}_X^{p,p}$$

Example 6.36. By the Poincaré-Lelong formula, $-log(\|s\|^2)$ is a Green current for $div(s)$.

Remark 6.37. The Green current is only defined up to the image of ∂ and $\overline{\partial}$. In its class we can always find a representative form which is smooth over $X - Z$ and with log singularities along Z.

From now we have seen nice properties of the complex of currents, but unlike the case of differential forms, some important properties are missing.

Theorem 6.38. $\mathcal{D}^*$ *can not been endowed with a nice ring structure. Similarly we can not define* f^* *in general.*

However, a product law can be defined for some particular currents, and the pullback can be defined for smooth morphisms, or under transversal intersection assumption.

Definition 6.39. If Z and W are cycles with Green currents g_Z and g_W such that $codim(Z \cap W) = codim(Z) + codim(W)$, we can define

$$g_Z \star g_W := g_Z \wedge \omega(g_W) + \delta_Z \wedge g_W \in \tilde{\mathcal{D}}^{p+q-1,p+q-1}$$

This definition seems quite unnatural but still satisfies some good properties.

Proposition 6.40. *Thus product is associative, commutative and it is a Green current for* $Z \cdot W$.

6.2.6 Arithmetic Chow group

We can now define an arithmetic analog to the usual Chow group.

Definition 6.41. $\widehat{Z}^p(\mathcal{X})$ is the free abelian group generated by classes (Z, g_Z) where g_Z is a Green current for $Z \in Z^p(X)$. We can distinguish the subgroup of rational cycles, $\widehat{Rat}^p(\mathcal{X})$ generated by cycles of the form $(div(f), -log(\|s\|^2))$ where f is a rational function on some (irreducible normal) subvariety $W \subset X$ of codimension $p-1$. Finally, the arithmetic Chow group is

$$\widehat{CH}^p(\mathcal{X}) := {}^{\widehat{Z}^p(\mathcal{X})}/_{\widehat{Rat}^p(\mathcal{X})}.$$

Unlike the classical case, the ring structure is not exactly well defined.

Proposition 6.42. *$\widehat{CH(\mathcal{X})}_{\mathbb{Q}}$ is a ring.*

Remark 6.43. We don't know how to define the intersection product over $\mathbb{Z}$; in some sense, if we had a resolution of singularities it should be possible.

This product can be express concretely in some special cases.

Proposition 6.44. *If Z and W intersect properly over $\mathbb{C}$, then $(Z, g_Z)\cdot(W, g_W) = (Z \cdot W, g_Z \star g_W))$. This class is supported over $Z \cap W$.*

We have an exact sequence

$$CH^{p,p-1}(X) \xrightarrow{Reg} \tilde{\mathcal{A}^{p,p}} \longrightarrow \widehat{CH}^p(\mathcal{X}) \longrightarrow CH^p(X) \longrightarrow 0$$

Where the first term is related to higher (classical) K-theory. Notes that the last surjectivity ensures that each cycle is equivalent to a cycle Z' which has a Green current. In fact the existence of a Green current for Z can be proved without the rational equivalence assumption.

Example 6.45. Over $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_K)$ (with K a number field) we have,

$$\mathcal{O}_X^* \longrightarrow \mathbb{R}^{r_1+r_2} \longrightarrow \widehat{CH}^1(\mathcal{X}) \longrightarrow Cl(\mathcal{O}_K) \longrightarrow 0$$

the class group is finite, and the quotient of $\mathbb{R}^{r_1+r_2}$ by the image of Reg is a cylinder. So we have an explicit description of this Chow group.

In particular over Spec($\mathbb{Z}$), the arithmetic Chow group is isomorphic to $\mathbb{R}$ via the morphism $\widehat{deg}$.

6.2.7 Pushforward

We have defined all the arithmetic analogs to K-theory and Chow groups, and we have the formalism of Chern classes, so we know how to construct the characters $\widehat{ch}(E)$ and $\widehat{Td}(E)$. The last objective is to construct a pushforward map. As in the geometrical case, it will be easy in the world of Chow group, and difficult in K-theory.

Definition 6.46. For $f : \mathcal{X} \to \mathcal{Y}$ proper and smooth of relative dimension e over the complex points, we have a pushforward map f_* such that

$$f_*(Z, g_Z) := \left(f_* Z, \int_f g_Z\right).$$

Here $\int_f$ means that we integrate along the fibers of f.

Example 6.47. For line bundle bundle we have

$$c_1(L, \|\cdot\|) = \left(div(s), -log(\|s\|^2)\right)$$

To define f_* in K-theory, we have to define a metric on the pushforward of a vector bundle, but in general it is just a coherent sheaf, so this is complicated. We will restrict to the case where f is smooth.

Axiomatic approach For a coherent sheaf $\mathcal{F}$ and

$$0 \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_n \longrightarrow \mathcal{F} \longrightarrow 0$$

a smooth resolution, a metric on $\mathcal{F}$ should be a metric on each term. So for $\overline{E}$ an hermitian bundle on X, we can define a metric on $\overline{T_f}$ and $\overline{f_* E}$.

By the Grothendieck-Riemann-Roch in the geometric case, $ch(\overline{f_* E}) - f_* \left(ch(\overline{E}) Td(\overline{T_f}) \right)$ is closed, hence by dd^c lemma it can be written

$$ch(\overline{f_* E}) - f_* \left(ch(\overline{E}) Td(\overline{T_f}) \right) = dd^c T(\overline{E}, \overline{T_f}, \overline{f_* E})$$

The emergence of this class T is similar to the existence of the Bott-Chern class. Unlike the latter, the class T is not unique. Yet we can classify all the solutions to this differential equation.

Proposition 6.48. *If $T(\overline{E}, \overline{T_f}, \overline{f_* E})$ and $T'(\overline{E}, \overline{T_f}, \overline{f_* E})$ are 2 solutions, there exist an additive characteristic class S such that*

$$T(\overline{E}, \overline{Tf}, \overline{f_* E}) - T'(\overline{E}, \overline{Tf}, \overline{f_* E}) = f_* \left(ch(E) \cdot Td(T_f) \cdot S(T_f) \right)$$

It it a relation in the space of differential forms.

Notation.

$$f_*^T(\overline{E}) = \overline{f_* E} - T(\overline{E}, \overline{Tf}, \overline{f_* E}).$$

We can now state the main result of these notes.

Theorem 6.49 (Grothendieck-Riemann-Roch-Arakelov)**.** *There is an unique T_0 such that the following diagram is commutative.*

$$\begin{tikzcd}
\hat{K}_0(X) \arrow[r, "f_*^{T_0}"] \arrow[d, "\widehat{ch}(\cdot)\widehat{Td}(T_X)"'] & \hat{K}_0(Y) \arrow[d, "\widehat{ch}(\cdot)\widehat{Td}(T_Y)"] \\
\widehat{CH}^*(X) \arrow[r, "f_*"'] & \widehat{CH}^*(Y)
\end{tikzcd}$$

Furthermore, if T^S is another solution we have

$$\widehat{ch}\left(f_*^{T^S}(\overline{E}) \right) = f_* \left(\widehat{ch}(\overline{E}) \widehat{Td}(\overline{T_f}) - S(T_f) Td(T_f) ch(E) \right).$$

The first part of the theorem may seem useless because T_0 is not explicit, but the second part patch this problem.

Concrete construction For a smooth proper map of complex varieties $X \to Y$, a metric on T_f induces a vertical Kähler form on the fibers X_y. To compute f_*E in K-theory we have to compute all the bundles $R^i f_* E$ (this sheaf is indeed a vector bundle due to the smoothness assumption), but locally the higher image is computed as a cohomology group on fibers ! So we need to find a natural metric on $H^i(X_y, E)$ (the smoothness assumption ensure that all the fibers are smooth and diffeomorphic).

Definition 6.50. For an holomorphic vector bundle E we can construct the bundle $\mathcal{A}_X^{0,q}(E) := \mathcal{A}_X^{0,q} \otimes E$ of $(0,q)$-differential forms with values in E.

For $\xi_1, \cdots, \xi_r$ a local frame of E, local sections of $\mathcal{A}_X^{0,q}(E)$ are

$$s = \sum_i \eta_i \xi_i,$$

for η_i a $(0,q)$-form on X.

In general we can not differentiate sections of E because we would have to define a connection. Yet the operator $\overline{\partial}$ is well defined. The following fact is well-known.

Proposition 6.51. *We can use the previous complex to compute the cohomology of the sheaf E*

$$H^q\left(X_y, (\mathcal{A}^{0,*}(E), \overline{\partial})\right) = H^q(X_y, E).$$

But now the bundle $\mathcal{A}^{0,q}(E)$ has a metric induced by the metric on E. More precisely we set

$$< s, s' >= \int_{X_y} \sum_{i,j} \eta_i \wedge \eta_j < \xi_i, \xi_j > \omega_Y^{e-q}$$

where the last term is the Kähler class on the fibers.

By Hodge theory we have the subspace of harmonic forms $\mathcal{H}^q \subset \mathcal{A}^{0,q}$ which is isomorphic to $H^q(X_y, E)$, so we find a natural metric on this space. However we also have to consider the L^2 metric, on the space of harmonic forms. The last one is not continuous in y, so we have to correct it.

Proposition 6.52 (Bismut-Köhler)**.** *We can see the higher analytic torsion as a correction to the L^2 metric. Then we obtain the desired metric on the cohomology of the fibers.*

Explicitly, the analytic torsion corresponds to the class

$$R(L) := \sum_{m\ odd} (\zeta'(-m) + \zeta(m)) \left(1 + \frac{1}{2} \cdots \frac{1}{m}\right) \frac{c_1(L)^m}{m!}$$

Proposition 6.53. *$det(f_*E)_y = \otimes_q \bigwedge^{top} H^q(X_y, E)^{\otimes(-1)^q}$.*

Example 6.54. On a complex

$$0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_{n-1} \longrightarrow V_n \longrightarrow 0$$

with a fixed norm on each term, we have $det(V_\bullet) = det(H(V_\bullet))$, but we want to compute the metric.
Let d^* be the adjoint of d and $\Delta = dd^* + d^*d$. Recall that $Ker(\Delta_i) =: \mathcal{H}_i \subset V_i$ is isomorphic to $H_i(V_\bullet)$, and it has an induced metric.

So to construct the true metric on $det(V_\bullet)$ we have to correct the L^2 metric on $det(H(V_\bullet))$ by some kind of $det(\Delta)$, but this is not trivial because Δ is an operator acting on infinite dimension space, so the definition of its determinant is unclear.

In fact, if $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ are the eigenvalues Δ, we can use the function

$$\zeta(s) = \sum_{i=0}^{+\infty} \frac{1}{\lambda_i^s}$$

which is holomorphic and can be continued to $s = 0$, to set $det'(\Delta) := \zeta(0)$.
Finally the correct metric is the Quillen metric given by

$$\|\cdot\|_Q = \|\cdot\|_{L^2} \prod_q exp(det'(\Delta_q))^{(-1)^q} q.$$

7 Schemes, adic spaces, $\mathbb{C}$-analytic spaces as analytic stacks, GAGA, duality - by Ferdinand Wagner

7.1 Schemes

We have a fully faithful functor from affine schemes to affine analytic stacks

$$F : \begin{cases} AffSch \hookrightarrow AffAnStacks \\ \mathrm{Spec}(R) \mapsto \mathrm{AnSpec}(R) \end{cases}$$

where AnSpec(R) is endowed with the trivial analytic ring structure. We will see that F extends to a fully faithful functor from schemes to analytic stacks as in the following diagram

$$\begin{tikzcd} Sch \arrow[r, hook] \arrow[d, equal] & AnStack \arrow[d, equal] \\ Sh_{Zar}(Aff) \arrow[r, dashed, "\phi"] & Sh_{!}(AffAnStack) \end{tikzcd} \tag{4}$$

where we recall that analytic stacks are sheaves over affine analytic stacks for the !-topology. To construct the dashed arrow ϕ, we have to show that Zariski covers are sent to !-covers.

Remark 7.1. The morphism $\mathrm{AnSpec}(R[\frac{1}{f}]) \hookrightarrow \mathrm{AnSpec}(R)$ is a closed immersion but is usually not open.

Definition 7.2 (Open immersion)**.** An open immersion is a monomorphism j such that j^* admits a fully faithful left adjoint $j_!$ and verifies the projection formula.

Definition 7.3 (Closed immersion)**.** A closed immersion is a proper morphism.

In particular, the morphism $\mathrm{AnSpec}(B) \to \mathrm{AnSpec}(A)$ is a closed immersion if and only if:

1. B has the induced analytic ring structure from A,
2. the diagonal $\Delta : \mathrm{AnSpec}(B) \to \mathrm{AnSpec}(B) \times_{\mathrm{AnSpec}(A)} \mathrm{AnSpec}(B)$ is an isomorphism.

Remark 7.4. The condition 2 is equivalent to B being idempotent over A, id est $B \otimes_A^L B \cong A$.

Remark 7.5. As a consequence, closed immersions of schemes are not sent to closed immersions of analytic stacks.

But still if

$$\bigsqcup_{i=1}^{n} \mathrm{Spec}\left(R\left[\frac{1}{f_i}\right]\right) \to \mathrm{Spec}(R)$$

is a Zariski cover, then

$$\bigsqcup_{i=1}^{n} \mathrm{AnSpec}\left(R\left[\frac{1}{f_i}\right]\right) \to \mathrm{AnSpec}(R)$$

is a !-cover.

To see it, we need to check:

$$R \in \left\langle \mathrm{im}\left(D\left(R\left[\frac{1}{f_i}\right]\right) \to D(R)\right) \mid i = 1, \dots, n \right\rangle$$

where we consider the category spanned by the image and closed under retracts, limits, colimits and tensor product. But we have

$$R \cong \left(\bigoplus_{1 \leqslant i \leqslant n} R\left[\frac{1}{f_i}\right] \to \bigoplus_{1 \leqslant i < j \leqslant n} R\left[\frac{1}{f_i f_j}\right] \to \dots \to R\left[\frac{1}{f_1 \dots f_n}\right]\right)$$

which concludes the proof.

Remark 7.6. We can replace Sh_{Zar} by $\omega - fpqc$ in Diagram 4.

Remark 7.7. Every map between affine schemes (and more generally every quasi-compact quasi-separated map of schemes) is sent to a proper map by the functor from schemes to analytic stacks. In other words, if we have a map $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ of affine schemes, it is sent to a map of analytic stacks $f : \mathrm{AnSpec}(S) \to \mathrm{AnSpec}(R)$ and we have

$$f_* = f_! : D(S) \to D(R)$$

whose right adjoint is $f^! = R\mathrm{Hom}(S, -) : D(R) \to D(S)$.

7.2 Adic spaces

Recall 7.8. Let M be in $D(R)$ and f in the underlying set $R(*)$ of R. We recall that P_R is $R[\mathbb{N} \cup \infty]/R[\infty]$. Let us recall a few definitions.

1. M is f-gaseous if the $(1-f)$-shift is an isomorphism of $\underline{\mathrm{Hom}}_R(P_R, M)$. Intuitively, it means that for every nullsequence $(m_n)_{n \in \mathbb{N}}$, the sum $\sum_{n=1}^{\infty} f^n m_n$ is well defined.

2. M is solid if 1-gaseous. Intuitively, it means that every nullsequence is summable.

3. M is f-solid if it is solid and f-gaseous. Intuitively, it means that $|f| \leqslant 1$.

Recall 7.9. $\mathrm{Spa}(R,R^+)$ is roughly the datum of the valuations $|\cdot|$ from R to an ordered group $\Gamma \cup \{0\}$ modulo equivalences such that $|\cdot|_{|R^+} \leqslant 1$.

Definition 7.10. We define $(R,R^+)_\square$ to be the analytic ring such that $(R,R^+)_\square^\triangleright$ is the solidification of R at all f in R^+ and whose analytic structure is given by

$$D((R,R^+)_\square) = \{M \in D(R) \mid \forall f \in R^+,\ M \text{ is } f\text{–solid}\}.$$

Remark 7.11. Thanks to Definition 7.10, we obtain a functor from adic sheafy spaces to analytic stacks which sends $\mathrm{Spa}(R,R^+)$ to $\mathrm{AnSpec}(R,R^+)_\square$.

Remark 7.12. There are two functors

$$\begin{cases} Sch & \rightrightarrows & adic\ Spaces \\ \mathrm{Spec}(R) & \mapsto & \mathrm{Spa}(R,\mathbb{Z}) \\ \mathrm{Spec}(R) & \mapsto & \mathrm{Spa}(R,R) \end{cases}$$

from schemes to adic spaces which give two functors from schemes to analytic stacks

$$\begin{cases} Sch & \to & \mathrm{AnStack} \\ X & \mapsto & (X,\mathbb{Z})_\square \\ X & \mapsto & (X,X)_\square := X_\square \end{cases}$$

from schemes to analytic stacks, where X^{alg} is $(X,\mathbb{Z})_\square = X^{alg} \times_{\mathrm{AnSpec}(\mathbb{Z})} \mathrm{AnSpec}(\mathbb{Z},\mathbb{Z})_\square$.

Remark 7.13. If

$$j : \mathrm{AnSpec}\left(R\left[\frac{1}{f}\right], R\left[\frac{1}{f}\right]\right)_\square \to \mathrm{AnSpec}(R,R)_\square$$

is an open immersion, we have

$$j^* = \left((-)\left[\frac{1}{f}\right]\right)^{L_{\square_{\frac{1}{f}}}} \cong (-)[T]^{L_{\square_T}}/(1-fT) = R\underline{\mathrm{Hom}}(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T],-)/(1-fT)$$

where we write $\square_f$ the solidification at f. We have

$$j_! = \left(\left(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T]\right)\ [-1] \otimes^L \mathbb{R}\right)/(1-fT) \otimes^L_{(R,R)_\square} (-).$$

7.3 $\mathbb{C}$-analytic spaces as analytic stacks

We want to describe the functor from Stein varieties to analytic stacks

$$X = \bigcup_{X \subset K \text{ compact Stein}} K \xrightarrow{(-)^{an}} \bigcup_{X \subset K \text{ compact Stein}} \mathrm{AnSpec}(O(K)^\dagger)$$

where $\mathcal{O}(K)^\dagger = \mathrm{colim}_{K \subset U}\, \mathcal{O}(U)$ is the ring of holomorphic overconvergent functions on K and has an analytic strucure induced by $\mathbb{C}_{\mathrm{gas}}$.

Example 7.14. Any

$$V(I) \subset \overline{\mathbb{D}^n} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| \leqslant 1\}$$

is compact Stein. We have

$$\mathcal{O}(\mathbb{D}^n)^\dagger_{\text{gas}} \cong \left(\mathcal{O}(\overline{\mathbb{D}^1})_{\text{gas}}\right)^{\otimes_{\mathbb{C}_{\text{gas}}} n}$$

and

$$\mathcal{O}(\overline{\mathbb{D}^1})^\dagger = \operatorname{colim}_{r>1} \left\{ \sum_{n=0}^{\infty} a_n T^n \mid a_n \in \mathbb{C},\, |a_n| r^n \xrightarrow{qexp} 0 \right\} = \operatorname{colim}_{r>1} \mathbb{C}_{\text{gas}} \left[\left(\frac{\widehat{T}}{r}\right)\right]$$

where qexp means that the convergence is quasi-exponential.

If we set

$$\mathcal{O}(S^1)^\dagger = \operatorname{colim}_{r>1} \left\{ \sum_{n=-\infty}^{\infty} a_n T^n \mid |a_{\pm n}| r^n \xrightarrow{qexp} 0 \right\}$$

we can construct the analytic stack $\mathbb{P}^{1,an}_{\mathbb{C}_{\text{gas}}}$ by gluing two copies of $\operatorname{AnSpec}(\mathcal{O}(\overline{\mathbb{D}^1})^\dagger_{\text{gas}})$ along $\operatorname{AnSpec}(\mathcal{O}(S^1)^\dagger_{\text{gas}})$.

Theorem 7.15. *The equality*

$$\mathbb{P}^{1,an}_{\mathbb{C}_{\text{gas}}} \cong \mathbb{P}^{1,alg}_{\mathbb{C}} \times_{\operatorname{AnSpec}(\mathbb{C})} \operatorname{AnSpec}(\mathbb{C}_{\text{gas}})$$

holds.

More generally, we have GAGA:

Theorem 7.16. *If X is a proper scheme over $\mathbb{C}$, then the equality*

$$X^{an} \cong X^{alg}_{\mathbb{C}_{\text{gas}}}$$

holds.

Remark 7.17. In particular, we have

$$D(X^{an}) \cong D(X^{alg}_{\mathbb{C}_{\text{gas}}})$$

which is GAGA for quasicoherent sheaves.

Remark 7.18. Classical GAGA can be deduced from this one but it is not straightforward.

Proof. Let us do a sketch of proof for Theorem 7.15. We work over $\operatorname{AnSpec}(\mathbb{C}_{\text{gas}}[T])$. Let us define

$$\mathcal{O}(\{|T| \leq 1\})^\dagger := \mathcal{O}(\overline{\mathbb{D}^1})^\dagger,$$
$$\mathcal{O}(\{|T| = 1\})^\dagger := \mathcal{O}(S^1)^\dagger$$

and

$$\mathcal{O}(\{|T| \geqslant 1\})^{\dagger} := \operatorname{colim}_{r>1} \left\{ \sum_{n=-m}^{\infty} a_n T^{-n} \mid |a_n| r^n \xrightarrow{qexp} 0 \right\}$$

where the last one can intuitively be seen as $\mathcal{O}(\{|T^{-1}| \leqslant 1\})^{\dagger}[(T^{-1})^{-1}]$.

Step 1 These are all idempotents algebras over $\mathbb{C}_{\text{gas}}[T]$ and the equality

$$\mathcal{O}(\{|T| \leqslant 1\})^{\dagger} \otimes^{L}_{\mathbb{C}_{\text{gas}}[T]} \mathcal{O}(\{|T| \geqslant 1\})^{\dagger} = \mathcal{O}(\{|T| = 1\})^{\dagger}$$

holds.
Step 2 The morphism

$$\operatorname{AnSpec}(\mathcal{O}(\{|T| = 1\})^{\dagger}) \sqcup \operatorname{AnSpec}(O(\{|T| \geqslant 1\})^{\dagger}) \to \operatorname{AnSpec}(\mathbb{C}_{\text{gas}}[T])$$

is a !-cover. To show it, we need $\mathbb{C}_{\text{gas}}[T]$ to be in $< im(\varphi_*) >$ where $\varphi : \mathbb{C}_{\text{gas}}[T] \to \mathcal{O}(\{|T| \leqslant 1\})$. It can be shown using Čech complex.
Steps 1+2 The equality

$$\operatorname{AnSpec}(\mathcal{O}(\{|T| \leqslant 1\})) \cup_{\operatorname{AnSpec}(\mathcal{O}(\{|T|=1\})^{\dagger})} \operatorname{AnSpec}(\mathcal{O}(\{|T| \geqslant 1\})^{\dagger}) = \operatorname{AnSpec}(\mathbb{C}_{\text{gas}}[T])$$

holds. ☐

Now, let us give some ideas used in the proof of Theorem 7.16.

Lemma 7.19. *If $f : X \to Y$ is a proper map of schemes, then the morphism*

$$X_{\square} \xrightarrow{\cong} X^{alg} \times_{Y^{alg}} Y_{\square}$$

is an isomorphism.

Proof. Let us give the proof of Lemma 7.19. Without loss of generality, we suppose $Y = \operatorname{Spec}(R)$. Then it is true by valuative criterion of properness. ☐

To obtain the result of Theorem 7.16, we should show that, for $\operatorname{Spec}(A)$ an open in X, adic spaces glued from $\operatorname{Spa}(A, A)$ are exactly adic spaces glued from $\operatorname{Spa}(A, R)$. The first ones are sent to $X_{\square}$ and the latter are sent to $X^{alg} \times_{\operatorname{AnSpec}(R)} \operatorname{AnSpec}(R, R)_{\square}$.

The general GAGA works similarly (see complex pdf lecture 6 of Clausen-Scholze).

7.4 Duality

For simplicity, let us stick to the case of schemes.

Theorem 7.20. *Let $f : X \to Y$ be a map of schemes.*

1. *If f is proper, then $f : X_{\square} \to Y_{\square}$ is proper.*

2. *If f is smooth, then $f : X_\square \to Y_\square$ is !-able and $f^! \mathcal{O}_Y = \omega_{X/Y}[d]$ where d is the relative dimension* $\dim(X/Y)$ *and $\omega_{X/Y}$ is the canonical line bundle* $\bigwedge^d \Omega^1_{X/Y}$.

Corollary 7.21. *Let $f : X \to Y$ be a map of schemes which is proper and smooth. We have*

$$R\operatorname{Hom}_Y(f_* F, \mathcal{O}_Y) = R\operatorname{Hom}_X(F, \omega_{X/Y}[d])$$

where $f_! F = f_* F$.

Remark 7.22. We keep notation of Corollary 7.21. If $Y = \operatorname{Spec}(k)$, we can apply H^{-i} to get

$$H^i(X, F)^\vee \cong \operatorname{Ext}^{d-i}(F, \omega_{X/Y}).$$

Proof. Let us give a sketch of proof of Theorem 7.20.

1. It follows by base change theorem.
2. We can reduce to $\mathbb{A}^1_Y \to Y$.

□

8 Betti stacks and analytic Riemann-Hilbert - by Adam Dauser

Recall:

1. X scheme $\mapsto X^{\mathrm{alg}}$ or $(X,\mathbb{Z})_\square$ or $(X,X)_\square$.
2. X adic space
3. X complex manifold, the associated analytic stack is build upon overconvergent functions on compact Stein.
4. GAGA: for X a proper scheme, $X \cong X^{\mathrm{alg}}_{\mathbb{C}_{\mathrm{gas}}}$

In this talk:

- {finite dim compact Hausdorff spaces} $\hookrightarrow$ AnStack. Moreover, quasicoherent sheaves will correspond to Betti sheaves. We embbed algebraic topology of compact Hausdorff spaces inside analytic stacks.
- There will be a surprising map $X \to \underline{X(\mathbb{C})}_{\mathrm{Betti}}$
- We will see how to characterize the maps to $\underline{X(\mathbb{C})}_{\mathrm{Betti}}$.

Definition 8.1. A proper map $f : \mathrm{AnSpec}(A) \to \mathrm{AnSpec}(B)$ is called descendable if

$$B \in \langle \mathrm{im} f_\star \rangle_{\text{finite limits, retracts, } \otimes}$$

We call f descendable of index $\leqslant m$ if for $\mathrm{Fib}(B \to A) = F$,

$$F^{\otimes_B m} \to A$$

is null-homotopic.

Remark 8.2. In practice, this is an easy condition to check, it will often come from studying Čech resolutions.

Example 8.3. If f is split, then f is descendable of index $\leqslant 1$.

Proposition 8.4. *Let $g_{i<j} : A_i \to A_j$ diagram of shape $(\mathbb{N}, \leqslant)$ in the category of analytic rings. If $f_i : A \to A_i$ is a proper descendable map of index $\leqslant m$. Then the induced map $f : A \to \mathrm{colim}_i A_i$ is descendable of index $\leqslant 2m$.*

Remark 8.5. In particular, if the maps $A \to A_i$ are split one can apply the proposition.

8.1 Betti stacks

Intuition from algebraic geometry, let k be a field:

- S finite set, $\mathrm{Spec}(\prod_{s\in S} k) = S$.
- S profinite set, $\mathrm{Spec}(\mathrm{Cont}(S,k)) = S$ where k has the discrete topology.

This induces a functor

$$\mathrm{Prof}^{\mathrm{light}} \to \mathrm{Sch} \to \mathrm{AnStack}$$

which maps a profinite S to $\mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{Z}))$.

Claim: this map is compatible with the Grothendieck topologies ! It maps covers to !-covers.

Proof. To check: if $S \to T$ is a surjective map of profinite sets, then

$$\mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{Z})) \to \mathrm{AnSpec}(\mathrm{Cont}(T,\mathbb{Z}))$$

is a !-cover. This assertion holds because the map can be written as a $(\mathbb{N},\leqslant)$-limit of split maps. ☐

Remark 8.6. AnSpec turns colimits in limits.

Corollary 8.7. *We obtain* $(-)_{\mathrm{Betti}} : \mathrm{Cond}^{\mathrm{light}} \to \mathrm{AnStack}$.

Proposition 8.8. *Let S be a finite dimensional compact Hausdorff space and $f : S' \to S$ a surjection from a light profinite set. Then $\mathbb{Z} \to Rf_\star\mathbb{Z} \in \mathcal{D}(S,\mathbb{Z})$ is descendable.*

Remark 8.9.

i) We only defined descendability for maps between analytic rings, but the notion makes sense in a more general context. In particular, it can be applied to ring objects in $\mathcal{D}(S,\mathbb{Z})$.

ii) A compact Hausdorff space is said to be of finite dimension if it has finite cohomological dimension. This is the case for finite dimensional CW complexes.

iii) $\mathcal{D}(S,\mathbb{Z})$ denotes the "usual" category of sheaves of (condensed) abelian groups on S.

iv) The last proposition is an input from algebraic topology, its proof has nothing to do with analytic rings.

Corollary 8.10. *For S a finite dimensional compact Hausdorff space, we have*

$$\mathcal{D}(\underline{S}_{\mathrm{Betti}}) \cong \mathcal{D}(S,\mathbb{Z}).$$

Proof. Let S be a light profinite, one has

$$\mathrm{QCoh}^{\mathrm{cond}}(\mathrm{Spec}(\mathrm{Cont}(S,\mathbb{Z}))) \cong \mathcal{D}(S,\mathbb{Z}).$$

To prove this, check that the amount of information to define both side is the same. In fact this statement about light profinite sets is still true if one considers only usual abelian groups and non condensed groups.

In general, choose a surjection $f : S_0 \to S$ with S_0 light profinite and take the Čech nerve $S_n = S_0 \times_S \cdots \times_S S_0$. Then for each n,

$$\mathcal{D}(\underline{S_n}_{\mathrm{Betti}}) \cong \mathcal{D}(S_n,\mathbb{Z}) \cong \mathrm{Mod}_{Rf_{n\star}\mathbb{Z}}\mathcal{D}(S,\mathbb{Z}).$$

Hence

$$\mathcal{D}(\underline{S}_{\mathrm{Betti}}) = \varprojlim_{n\in\Delta}\mathcal{D}(\underline{S_n}_{\mathrm{Betti}}) = \varprojlim_{n\in\Delta}\mathrm{Mod}_{Rf_{n\star}\mathbb{Z}}\mathcal{D}(S,\mathbb{Z}) = \mathcal{D}(S,\mathbb{Z}),$$

where the first equality comes from the !-cover, and the last comes from descendability of

$$\mathbb{Z} \to Rf_\star\mathbb{Z}.$$

□

The following theorem will be useful to construct maps to Betti stacks

Theorem 8.11 (Tannakian reconstruction of Betti stacks)**.** *Let S be a finite dimension compact Hausdorff space. For analytic stack X, there is an isomorphism*

$$\{\mathit{Maps}\ X \to \underline{S}_{\mathrm{Betti}}\} \cong \left\{\begin{matrix}\textit{Collections of idempotent algebras } A_Z\in\mathcal{D}(X) \textit{ for every closed subset } Z\subset S \\ \textit{such that } A_{\cap_{i\in I}Z_i}=\mathrm{colim}\,A_{Z_i} \textit{ and such that } \exists\{X_i\to X\} \textit{ by affine analytic stacks and} \\ f_i^\star A_Z \textit{ is connective+ for any finite union } Z=\cup Z_i, A_Z \textit{ is computed via the Čech complex of } A_{Z_i}\end{matrix}\right\}$$

Example 8.12.

i) Let X be a complex analytic manifold, the map $X \to \underline{X(\mathbb{C})}_{\mathrm{Betti}}$ is defined by the association

$$K \text{ compact Stein } \mapsto \mathcal{O}^\dagger(K) \in \mathcal{D}(X).$$

ii) The map $\arg : \mathbb{G}_{m,\mathbb{C}}^{\mathrm{alg}} \to \underline{S^1}_{\mathrm{Betti}}$ is given by sending a closed interval I to

$$\{f \in \mathcal{O}(S_I) \mid f \text{ has polynomial growth at infinity and at } 0\},$$

where S_I is the preimage of I by $\arg : \mathbb{C}^\star \to S^1$.

Lemma 8.13. $X \to \underline{X(\mathbb{C})}_{\mathrm{Betti}}$ *is a !-cover.*

We will now wonder what equivalence relation on X gives rise to $X(\mathbb{C})_{\mathrm{Betti}}$

Definition 8.14. Consider the overconvergent neighbourhood of the diagonal $\mathcal{O}^\dagger(\Delta) \subset X \times X$ as an equivalence relation. The quotient of X by the above equivalence relation is denoted $X^{\text{an-dR}}$.

Theorem 8.15 (Analytic Riemann-Hilbert)**.** *The map $X \to \underline{X(\mathbb{C})}_{\text{Betti}}$ factors through $X^{\text{an-dR}}$ and gives rise to an isomorphism.*

Proof. We need to compute $X \times_{X_{\text{Betti}}} X \to X \times X$. This identifies with the overconvergent neighbourhood of the diagonal:

Let $Z \subset X$ be a compact Stein. The equality

$$(Z \times X)^\dagger = X \times_{\underline{X(\mathbb{C})}_{\text{Betti}}} Z_{\text{Betti}}$$

follows by definition from Tannakian reconstruction. ☐

8.2 Riemann–Hilbert

Let X be a smooth complex variety. The functor

$$\text{Mod}^{\text{rh}}_{\mathcal{D}_X} \to \text{Perv}(X)$$

sends differential equations to the sheaf of solutions.

$\text{Perv}(X)$ is a subcategory of $\mathcal{D}(X(\mathbb{C}), \mathbb{C})$. For example, one can consider the differential equation of the logarithm.

Theorem 8.16. $\text{Mod}_{\mathcal{D}_X}\text{QCoh}(X) = \text{QCoh}(X^{\text{dR}})$, *where X^{dR} is the de Rham stack, given by $X/\sim$.*

In any case $\mathcal{D}(X^{\text{an-dR}}) \subset \text{Mod}_{\mathcal{D}_X}\mathcal{D}(X)$ where one cuts out by growth condition.

Corollary 8.17. $\mathcal{D}(X^{\text{an-dR}}) \cong \mathcal{D}(X, \mathbb{C}_{\text{gas}})$

Caveat: this equivalence does not involve regular holonomic $\mathcal{D}$-modules nor perverse sheaves. One can retreive the usual equivalence up to some fudging, but it is nontrivial.

9 A-schemes, formal-analytic surfaces and θ-invariants - by François Charles

9.1 Introduction

This lecture is divided in two parts : the first part aims to define objects obtained by gluing together arithmetic and analytic data, namely the A-schemes, and studying their relevance in diophantine geometry. The second part will develop a cohomological theory of quasi-coherent sheaves on them through the study of euclidean lattices, possibly of infinite rank, and give them an extra structure (nuclear Frechet spaces), as well as giving some applications.

9.1.1 Goals

The theory of A-schemes has multiple objectives :

- We would like to have a setting letting us do Arakelov geometry that is as flexible as scheme theory, allowing for example non-reduced or singular objects.
- We want to allow the use of singular metrics.
- It gives a setting to derive some classical algebraization theorems ; for instance (Borel, Pólya-Szegö, André, Bost, ...) give a criterion for a power series with integral coefficients to be an algebraic function.
- It allows us to bound the solution sets for algebraic differential equations (some efficient holonomy bounds were obtained by Calegari-Dimitrov-Tang in 2021)
- It helps to understand the set of integral points of an affine scheme over Spec($\mathbb{Z}$) that lie in a given compact subset.
- It allows to study the positivity of hermitian line bundles via the geometry of their total space.

9.1.2 Open questions

Some questions remain unanswered, for instance :

- This theory has some connections with the theory of analytic stacks ; how exactly are they related ?
- How can we study the positivity of vector bundles in Arakelov geometry ?
- What is its relationship with arithmetic intersection theory, with Riemann-Roch theorem ? (See the work of Dorian Ni for some elements of answer)
- Can we work out part of intersection theory on Shimura varieties via these methods ?

9.1.3 References

This talk uses as a reference for theta functions (Banaszczyk 80'), (J.B Bost 2015), (J.B Bost & F. Charles 2022, *Quasi-projective and formal analytic arithmetic surfaces*) and (J.B Bost & F. Charles 2024, *Infinite Dimensional Geometry of Numbers : Hermitian Quasi-coherent Sheaves and Theta Finiteness*).

9.2 Formal-analytic arithmetic surfaces and A-schemes

9.2.1 Geometric analogy

First of all, we give an analogy to understand the context of this lecture : suppose S is a complex analytic surface, take C a compact Riemann surface lying inside S (i.e. a closed subscheme of codimension 1). Then we can construct the normal bundle $N_C S$, which is a line bundle over C. There are two important cases :

- if $\deg(N_C S) < 0$, then the complex curve C can be contracted on a single point in S, and in this case there are many holomorphic functions defined in a neighborhood of C in S.
- if $\deg(N_C S) > 0$, i.e. the line bundle $N_C S$ is ample, then the situation is algebraic (by this we mean that every function defined in a neighborhood of C in S is algebraic).

We would like to do a similar observation for another type of surface. This is where the formal-analytic arithmetic surfaces come into play : the idea is to replace C by the scheme $\mathrm{Spec}(\mathcal{O}_K)$ for K a number field.

9.2.2 Formal-analytic arithmetic surfaces

Definition 9.1. A *formal-analytic arithmetic surface* over $\mathcal{O}_K$ is a pair $\tilde{\mathcal{V}} = (\hat{\mathcal{V}}, (V_\sigma, P_\sigma, \iota_\sigma)_{\sigma:K\hookrightarrow\mathbb{C}})$ with $\hat{\mathcal{V}}$ a formal scheme of dimension 2 over $\mathcal{O}_K$, $\hat{\mathcal{V}} \to \mathrm{Spec}(\mathcal{O}_K)$ smooth, and $|\hat{\mathcal{V}}| \xrightarrow{\sim} \mathrm{Spec}(\mathcal{O}_K)$ as topological spaces (locally, $\mathrm{Spf}(\mathcal{O}_K[[X]]) \to \mathrm{Spec}(\mathcal{O}_K)$) ; and for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, V_σ is a compact Riemann surface with a smooth non-empty boundary, P_σ a point in the interior of V_σ, and $\iota_\sigma : \hat{\mathcal{V}} \times_{K,\sigma} \mathrm{Spec}(\mathbb{C}) \xrightarrow{\sim} (V_\sigma)\hat{}_{P_\sigma}$ (the completion at P_σ), with compatibility conditions with respect to the complex conjugation.

One way to construct such a formal-analytic arithmetic surface $\tilde{\mathcal{V}}$ is to complete an arithmetic surface along an integral point (by choosing an integral point, taking an universal cover and possibly removing some points, then gluing). We can define vector bundles and hermitian vector bundles on $\tilde{\mathcal{V}}$:

Definition 9.2. An *hermitian vector bundle* on $\tilde{\mathcal{V}}$ is a pair $\tilde{\mathcal{E}} = (\hat{\mathcal{E}}, (E_\sigma, \varphi_\sigma, \|\cdot\|_\sigma)_{\sigma:K\hookrightarrow\mathbb{C}})$ with $\hat{\mathcal{E}}$ a vector bundle on $\hat{\mathcal{V}}$, E_σ a vector bundle on V_σ, $\|\cdot\|_\sigma$ an hermitian metric and φ_σ an isomorphism on $\hat{\mathcal{V}}$ such that $\varphi_\sigma : \hat{\mathcal{E}} \times_{K,\sigma} \mathrm{Spec}(\mathbb{C}) \xrightarrow{\sim} \iota_\sigma^* E_\sigma$, compatible with the complex conjugation.

Definition 9.3. We define the *global sections* of $\tilde{\mathcal{E}}$ on $\tilde{\mathcal{V}}$ as $\Gamma(\tilde{\mathcal{V}}, \tilde{\mathcal{E}}) = \{(\hat{s}, (s_\sigma)_{\sigma : K \hookrightarrow \mathbb{C}}\}$ with $\hat{s}$ a global section of $\hat{\mathcal{E}}$ and $s_\sigma \in \Gamma(V_\sigma, E_\sigma)$ extended holomorphically from $\hat{s}$ with an overconvergent condition, compatible with φ_σ.

Remark 9.4. If we suppose $\mathcal{E}$ has a metric, in a way $\Gamma(\tilde{\mathcal{V}}, \mathcal{E})$ comes from a more complicated object $\pi_* \tilde{\mathcal{E}} = (\Gamma(\hat{\mathcal{V}}, \hat{\mathcal{E}}), \prod_\sigma \Gamma(V_\sigma, \mathcal{E}_\sigma)$ with compatibility conditions), a pro-hermitian vector bundle.

9.2.3 The case where $\mathcal{O}_K = \mathbb{Z}$

If we suppose $\mathcal{O}_K = \mathbb{Z}$, then $\hat{\mathcal{V}} = \mathrm{Spf}(\mathbb{Z}[[X]])$. So we have $\tilde{\mathcal{V}} = (\hat{\mathcal{V}}, (V, P, \iota))$ and the completion $\hat{V}_P = \mathrm{Spf}(\mathbb{C}[[z]])$. It gives

$$\iota : \mathrm{Spf}(\mathbb{C}[[\mathrm{X}]]) \tilde{\rightarrow} \mathrm{Spf}(\mathbb{C}[[z]])$$

$$\psi = \iota^* z \hookleftarrow z.$$

In this special case it is enough to study $\psi \in \mathbb{R}[[X]]$ such that $\psi(0) = 0$ and $\psi'(0) \neq 0$. We may consider $\mathcal{O}(\tilde{\mathcal{V}}) = \Gamma(\tilde{\mathcal{V}}, \mathcal{O}_{\tilde{\mathcal{V}}}) = \{(\hat{\alpha} \in \mathbb{Z}[[X]], \alpha^{an} \in \mathcal{O}(V))$ such that $\alpha^{an} = \hat{\alpha} \circ \varphi\}$.

Example 9.5. $\tilde{\mathcal{V}} = (\mathrm{Spf}(\mathbb{Z}[[\mathrm{X}]], (\bar{D}(0,1), P = 0, \psi = \frac{X}{r})) \simeq (\mathrm{Spf}(\mathbb{Z}[[X]]), (\bar{D}(0,r), P = 0, \psi = X)$

We also have meromorphic functions on $\tilde{\mathcal{V}}$, whose set is denoted as $\Omega(\tilde{\mathcal{V}})$. For a specific example, consider $\tilde{\mathcal{V}} = \tilde{B}(r) := (\mathrm{Spf}(\mathbb{Z}[[X]], (\bar{D}(0,r), P = 0, \mathrm{can.}))$, where can. designates the canonical isomorphism. Then :

Theorem 9.6. *We have* $\mathcal{O}(\tilde{B}(r)) = \mathbb{Z}[X]$ *if* $r \geq 1$*. Otherwise,* $\mathcal{O}(\tilde{B}(r))$ *is much bigger.*

There is a better result :

Theorem 9.7 (Borel)**.** *If* $r \geq 1$*, then* $\Omega(\tilde{B}(r)) = \mathbb{Z}(X)$*.*

More generally, given a power series $\psi \in \mathbb{R}[[X]]$ with $\psi(0) = 0$ and $\psi'(0) \neq 0$, there exists a formal-analytic arithmetic surface $\tilde{\mathcal{V}} = \tilde{\mathcal{V}}(\psi) := (\mathrm{Spf}(\mathbb{Z}[[X]], (\bar{D}(0,1), P = 0, \psi))$. With these notations, if $|\psi'(0)| > 1$, then $\mathcal{O}(\tilde{\mathcal{V}}(\psi))$ is very large ; and if $|\psi'(0)| \leq 1$ (we say that ψ is *generic*), then $\mathcal{O}(\tilde{\mathcal{V}}(\psi)) = \mathbb{Z}$.

9.2.4 Back to the general case

Let $\tilde{\mathcal{V}} = (\hat{\mathcal{V}}, (V_\sigma, P_\sigma, \iota_\sigma)_{\sigma : K \hookrightarrow \mathbb{C}})$ be a formal-analytic arithmetic surface over $\mathcal{O}_K$. Associated to it, there is a normal bundle $\bar{N_P} \tilde{\mathcal{V}}$, it is an hermitian line bundle on $\mathrm{Spec}(\mathcal{O}_K)$, roughly giving the size of $\tilde{\mathcal{V}}$ "from the perspective of P". This hermitian line bundle has an Arakelov degree ; the main result is the following :

Theorem 9.8. *If* $\hat{\deg} \bar{N_P} \tilde{\mathcal{V}} > 0$*, then* $\mathcal{O}(\tilde{\mathcal{V}})$ *is a* $\mathbb{Z}$*-algebra of finite type and of transcendence degree* ≤ 1*. Moreover, any map from* $\tilde{\mathcal{V}}$ *to a projective scheme has an algebraic image.*

This result finds applications in relation with fundamental groups, and irrationality results via holonomy bounds.

10 The gaseous base stack and the stack of norms - by Ferdinand Wagner

In today's talk we present an analytic stack mixing archimedean and non-archimedean geometry: the hope is that this becomes an important object in the theory of Arakelov geometry.

10.1 Norms on analytic stacks

When we hear *norm*, we probably think of a map $|\cdot| : A \to \mathbb{R}_{\geq 0}$. In the context of analytic stacks, this is not the right intuition to have. Instead, we should think of the notion of "disks of radius r" $\overline{\mathbb{D}}^{\dagger}(r) \subset \mathbb{A}^1_A$ for every $r \in \mathbb{R}_{\geq 0}$. These disks will be closed and automatically overconvergent, as the notation suggests. In particular, norms in this sense can be pulled back along morphisms $\mathrm{AnSpec}(B) \to \mathrm{AnSpec}(A)$, contrarily to usual norms.

Let A be an analytic ring. We set $\mathbb{P}^1_A \coloneqq \mathbb{P}^{1,\mathrm{alg}}_{\mathbb{Z}} \times_{\mathrm{AnSpec}(\mathbb{Z})} \mathrm{AnSpec}(A)$. A norm on A is a morphism $N : \mathbb{P}^1_A \to [0,\infty]_{\mathrm{Betti}}$ satisfying certain conditions. These conditions will contain the informations that, in a sense, we have "$N(0) = 0$", "$N(x^{-1}) = N(x)^{-1}$", "$N(x \cdot y) = N(x) \cdot N(y)$". To make these conditions precise, we use the following

Remark 10.1. Every $f \in A$ identifies with a splitting $f : \mathrm{AnSpec}(A) \to \mathbb{P}^1_A$ of the map $\mathbb{P}^1_A \to \mathrm{AnSpec}(A)$. By composing it with $N : \mathbb{P}^1_A \to [0,\infty]_{\mathrm{Betti}}$, we obtain a morphism of analytic stacks $\mathrm{AnSpec}(A) \to [0,\infty]_{\mathrm{Betti}}$, that we call $N(f)$.

Definition 10.2. Let A be an analytic ring. A norm on A is a morphism $N : \mathbb{P}^1_A \to [0,\infty]_{\mathrm{Betti}}$ such that the following conditions hold

1. ("$N(0) = 0$") The morphism $N(0) : \mathrm{AnSpec}(A) \to [0,\infty]_{\mathrm{Betti}}$ factors through $\{0\}_{\mathrm{Betti}}$.
2. ("$N(x^{-1}) = N(x)^{-1}$") We have the following commutative diagram

$$\begin{tikzcd}
\mathbb{P}^1_A \arrow[r, "N"] \arrow[d, "(T \mapsto T^{-1})"] & {[0,\infty]_{\mathrm{Betti}}} \arrow[d, "(-)^{-1}"] \\
\mathbb{P}^1_A \arrow[r, "N"] & {[0,\infty]_{\mathrm{Betti}}.}
\end{tikzcd}$$

3. ("$N(x \cdot y) = N(x) \cdot N(y)$") We set $\mathbb{A}^{1,\mathrm{an}}_A \coloneqq N^{-1}([0,\infty))$. Let us consider the morphism

$$\mu : \mathbb{A}^{1,\mathrm{an}}_A \times \mathbb{A}^{1,\mathrm{an}}_A \to \mathbb{A}^1_A, \quad (T_1, T_2) \mapsto T_1 \cdot T_2,$$

which factors through $\mathbb{A}^{1,\mathrm{an}}_A$ by conditions 1 and 2. We have a commutative diagram

$$\begin{tikzcd}
\mathbb{A}^{1,\mathrm{an}}_A \times \mathbb{A}^{1,\mathrm{an}}_A \arrow[r, "N\times N"] \arrow[d, "\mu"] & {[0,\infty)_{\mathrm{Betti}} \times [0,\infty)_{\mathrm{Betti}}} \arrow[d, "\mathrm{mult.}"] \\
\mathbb{A}^{1,\mathrm{an}} \arrow[r, "N"] & {[0,\infty)_{\mathrm{Betti}}}.
\end{tikzcd}$$

4. Let $\mathbb{D}_A \coloneqq \mathrm{AnSpec}(A[\hat{T}])$ with the induced analytic ring structure, which has a canonical map $\varphi : \mathbb{D}_A \to \mathbb{P}^1_A$. Then we have a factorisation

$$\begin{tikzcd}
\mathbb{D}_A \arrow[r, "\varphi"] \arrow[rrd, dashed] & \mathbb{P}^1_A \arrow[r, "N"] & {[0,\infty]_{\mathrm{Betti}}} \\
 & & {[0,1]_{\mathrm{Betti}}} \arrow[u, hook]
\end{tikzcd}$$

Moreover, φ is an isomorphism over $[0,1)_{\mathrm{Betti}}$.

Here are remarks/explanations about this definition.

Remark 10.3. Conditions 1 and 2 imply that we have "$N(\infty) = \infty$" and "$N(1) = 1$". In particular, the morphism μ in condition 3 factors through $\mathbb{A}^{1,\mathrm{an}}_A$.

Remark 10.4. Condition 3 implicitly uses the fact that we have $\mathbb{A}^{1,\mathrm{an}}_A \subseteq \mathbb{A}^1_A$. Indeed, by the condition "$N(\infty) = \infty$", we obtain that the structure sheaf of the divisor at ∞, i.e. $\mathcal{O}_{\infty\subset\mathbb{P}^1_A} \coloneqq \mathcal{O}_{\mathbb{P}^1_A|T^{-1}}$, is an algebra over $\mathcal{O}_{N^{-1}(\{\infty\})}$. Thus we have $\mathcal{O}_{\infty\subset\mathbb{P}^1_A}|_{\mathbb{A}^{1,\mathrm{an}}_A} = 0$. This implies that T^{-1} is invertible on $\mathbb{A}^{1,\mathrm{an}}_A$, and thus we have $\mathbb{A}^{1,\mathrm{an}}_A \subseteq \mathbb{A}^1_A$.

Remark 10.5. In the conditions we just presented we have commutative diagrams, which in the higher-categorical setting would translate in higher coherency additional data. Fortunately, mapping spaces from an analytic stack to Betti stacks are just sets. Thus commutativity of those diagrams are just properties, and not extra data.

Remark 10.6. Condition 4 tells us that $\mathbb{D}_A$ sits between the open and the closed unit disk,

Remark 10.7. We have no triangular inequality condition, i.e. no condition "$N(x+y) \leq N(x) + N(x)$".

More generally, we can define what a norm on an analytic stack is by glueing from the affine case.

Definition 10.8. Let $\mathfrak{N}$ be the universal normal analytic stack, i.e. such that we have

$$\{\text{norms on } A\} \simeq \{\text{maps } \mathrm{AnSpec}(A) \to \mathfrak{N}\}.$$

$\mathfrak{N}$ is called the *stack of norms.*

Remark 10.9. $\mathfrak{N}$ exists by definition. Indeed, we can define a sheaf on affine analytic stacks exactly in this way, by $\mathfrak{N}(A) \coloneqq \{\text{norms on } A\}$ and then check that it satisfies descent.

Definition 10.10. We define the anima of norms on an analytic stack X as $\mathrm{Hom}_{\mathrm{AnStack}}(X, \mathfrak{N})$.

10.2 How to construct norms

We recall that if S is locally compact Hausdorff of finite cohomological dimension and X is an analytic stack, we have an isomorphism

$$\{\text{maps } X \to S_{\mathrm{Betti}}\} \simeq \left\{ \begin{array}{c} \text{collection of idempotents } A_Z \in \mathrm{D}(X) \text{ for all closed } Z \subseteq S \\ + \text{ compatibility condition for intersections and finite unions} \\ + \text{ connectivity condition} \end{array} \right\}$$

If we have $f : X \to S_{\mathrm{Betti}}$, under this identification we have $A_Z \coloneqq \mathcal{O}_{f^{-1}(Z)}$.

In particular, to construct a norm $N : \mathbb{P}^1_A \to [0,\infty]_{\mathrm{Betti}}$ it suffices to specify idempotents $\mathcal{O}(\{|T| \le r\})^\dagger \coloneqq \mathcal{O}_{N^{-1}([0,r])}$ for all $r \in \mathbb{R}_{\ge 0}$. All the other idempotent algebras are determined by the compatibility conditions with union, intersection, plus condition 2 of Definition 10.2. Moreover, the conditions 1 to 4 of Definition 10.2 can be restated in terms of these idempotent algebras. For example, condition 3 tells us that we have a morphism $\mathcal{O}(\{|T| \le r_1 \cdot r_2\})^\dagger \to \mathcal{O}(\{|T| \le r_1\}) \otimes_A \mathcal{O}(\{|T| \le r_2\})$ sending T to $T_1 \otimes T_2$.

Remark 10.11 (Overconvergency is necessary)**.** We have

$$\mathbb{Z}_{[0,r]} \simeq \operatorname*{colim}_{r' > r} \mathbb{Z}_{[0,r']},$$

hence we must have

$$\mathcal{O}_{N^{-1}([0,r])} \simeq \operatorname*{colim}_{r' > r} \mathcal{O}_{N^{-1}([0,r'])}.$$

Thus $\mathcal{O}(\{|T| \le r\})^\dagger \coloneqq \mathcal{O}_{N^{-1}([0,r])}$ is automatically a ring of overconvergent functions. This means that whatever notion of closed disk we choose , the "regular functions" on it must be overconvergent. For example, over $\mathbb{Q}_{p\square}$, the Tate algebras are idempotent but they do not define a norm, since they are not compatible with taking limits. We should instead consider an overconvergent version of them.

Corollary 10.12. *There exist norms on* $\mathbb{Q}_{p\square}, \mathbb{R}_{\mathrm{gas}}, \mathbb{C}_{\mathrm{gas}}$ *such that* $\mathcal{O}(\{|T| \le r\})^\dagger$ *is the usual overconvergent algebra.*

Corollary 10.13. *For every* $\lambda \in (0,1)$ *there is a unique norm on* $\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}}$ *such that we have "*$N(q) = \lambda$*". Moreover, these norms combine to a unique norm on* $\mathrm{AnSpec}(\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}}) \times (0,1)$.

Proof sketch. Replace $\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}}$ by $A \coloneqq \mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}}[q^{\frac{1}{n}} \,|\, n \geq 1]$. For the existence, we define

$$\mathcal{O}(\{|T| \leq r\})^{\dagger} \coloneqq \underset{\substack{\alpha\in\mathbb{Q} \text{ s.t.}\\ \lambda^{\alpha}>r}}{\operatorname{colim}} A[\widehat{q^{-\alpha}T}].$$

For the uniqueness, we suppose that we have a norm on $\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}}$ such that "$N(q) = \lambda$". We fix r and consider $\alpha \in \mathbb{Q}$ such that $\lambda^{\alpha} > r$. By hypothesis, we have "$N(q^{-\alpha}) = \lambda^{-\alpha}$", thus we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{AnSpec}(A) \times \mathrm{AnSpec}(\mathcal{O}(\{|T| \leq r\})^{\dagger} \arrow[r, "N\times N"] \arrow[d, "q^{-\alpha}\times\text{incl.}"] & \{\lambda^{-\alpha}\} \times [0,r] \arrow[d, "\text{mult.}"] \\
\mathbb{A}^{1,\mathrm{an}}_A \times \mathbb{A}^{1,\mathrm{an}}_A \arrow[d, "\mu"] & {[0, \lambda^{-\alpha} r]} \arrow[d] \\
\mathbb{A}^{1,\mathrm{an}}_A \arrow[r, "N"] & {[0,\infty].}
\end{tikzcd}$$

Since we have $\lambda^{-\alpha} r < 1$, the morphism $\mathrm{AnSpec}(A) \times \mathrm{AnSpec}(\mathcal{O}(\{|T| \leq r\}) \to [0,\infty]$ coming from this diagram factors through $[0,1)$. By condition 4 of Definition 10.2, the composition $\mathbb{D}_A = \mathrm{AnSpec}(A[\widehat{q^{-\alpha}T}]) \to \mathbb{A}^{1,\mathrm{an}}_A \to [0,\infty]$ factors over $[0,1]$ and becomes an isomorphism over $[0,1)$. Consequently, the map $\mathrm{AnSpec}(A) \times \mathrm{AnSpec}(\mathcal{O}(\{|T| \leq r\})^{\dagger}) \to \mathbb{A}^{1,\mathrm{an}}_A$ factors over $\mathbb{D}_A$, and we get a map

$$A[\widehat{q^{-\alpha}T}] \to \mathcal{O}(\{|T| \leq r\})^{\dagger}.$$

We pass to the colimit and we get a map

$$\underset{\substack{\alpha\in\mathbb{Q} \text{ s.t.}\\ \lambda^{\alpha}>r}}{\operatorname{colim}} A[\widehat{q^{-\alpha}T}] \to \mathcal{O}(\{|T| \leq r\})^{\dagger}.$$

To show that it is an isomorphism, we use the fact that

$$\mathcal{O}(\{|T| \leq r\})^{\dagger} \otimes^{L}_{A[T]} \mathcal{O}(\{|T| \leq r'\})^{\dagger} = 0 \qquad \text{if } r' > r$$

and that the colimit in the left-hand side is determined by this property (i.e. that tensoring it with $\mathcal{O}(\{|T| \leq r'\})^{\dagger}$ for $r' > r$ gives 0).

For the original analytic ring $\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}}$ the argument is quite the same but with some more difficulties. □

This unique norm gives a map $\mathrm{AnSpec}(\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}}) \times (0,1) \to \mathfrak{N}$.

Lemma 10.14. *Let A be any normed analytic ring and $\lambda \in (0,1)$. Then there exists a !-cover* $\mathrm{AnSpec}(B) \to \mathrm{AnSpec}(A)$ *such that there exists $q \in B(*)$ with "$N(q) = \lambda$". In particular, the morphism* $\mathrm{AnSpec}(\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}}) \times \{\lambda\} \to \mathfrak{N}$ *is a !-cover.*

Proof sketch. We will show that $\mathcal{O}(\{|T| = \lambda\})^\dagger \coloneqq \mathcal{O}_{N^{-1}(\{\lambda\})}$ receives a map from A which is a !-cover. Since the analytic ring structure on $\mathcal{O}_{N^{-1}(\{\lambda\})}$ is the induced one, the morphism $A \to \mathcal{O}_{N^{-1}(\{\lambda\})}$ is proper. Thus it is enough to show that we have

$$A \in \langle \mathrm{im}(\mathrm{D}(\mathcal{O}(\{|T| = \lambda\})^\dagger) \to \mathrm{D}(A) \rangle_{\text{fin. limits, retracts, } \otimes}.$$

Let us consider $\pi_* : \mathrm{D}(\mathbb{P}^1_A) \to \mathrm{D}(A)$. We have the following pullback diagram in $\mathrm{D}(\mathbb{P}^1_A)$

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^1_A} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{O}_{N^{-1}([\lambda,\infty])} \arrow[d] \\
\mathcal{O}_{N^{-1}([0,\lambda])} \arrow[r] & \mathcal{O}_{N^{-1}(\{\lambda\})}
\end{tikzcd}$$

In stable ∞-categories, pullbacks are also pushouts, so when we apply π_* we get a pushout diagram in $\mathrm{D}(A)$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \pi_* \mathcal{O}_{N^{-1}([\lambda,\infty])} \arrow[d] \\
\pi_* \mathcal{O}_{N^{-1}([0,\lambda])} \arrow[r] & \mathcal{O}(\{|T| = \lambda\})^\dagger. \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

We now show thanks to this pushout diagram, that we have an induced map $\mathcal{O}(\{|T| = \lambda\})^\dagger \to A$. To do this, we observe that we have maps

$$\mathcal{O}_{N^{-1}([0,\lambda])} \to \mathcal{O}_{0 \subset \mathbb{P}^1_A}, \qquad \mathcal{O}_{N^{-1}([\lambda,\infty])} \to \mathcal{O}_{\infty \subset \mathbb{P}^1_A}$$

Applying π_* gives the desired morphisms

$$\pi_* \mathcal{O}_{N^{-1}([0,\lambda])} \to A, \qquad \pi_* \mathcal{O}_{N^{-1}([\lambda,\infty])} \to A,$$

inducing by pushout the map $\mathcal{O}(\{|T| = \lambda\})^\dagger \to A$. ☐

Theorem 10.15. *For any analytic ring A we have*

$$\{(\text{norm } N \text{ on } A,\ q \in A(*)) \text{ s.t. ``} N(q) \subseteq (0,1)\text{''}\} \simeq \{\text{maps } \mathrm{AnSpec}(A) \to \mathrm{AnSpec}(\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}}) \times (0,1)\}$$

Proof sketch.

- Given (N, a), we consider the map

  $$(a, N(a)) : \mathrm{AnSpec}(A) \to \mathrm{AnSpec}(\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}}) \times (0,1).$$

  Here $a : \mathrm{AnSpec}(A) \to \mathrm{AnSpec}(\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}})$ is induced by $q \mapsto a$.

- Given a map $\mathrm{AnSpec}(A) \to \mathrm{AnSpec}(\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}}) \times (0,1)$, we associate to it the couple (N, a), where N is the pullback of the universal norm on $\mathrm{AnSpec}(\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}}) \times (0,1)$ and a is the image of q in $A(*)$.

☐

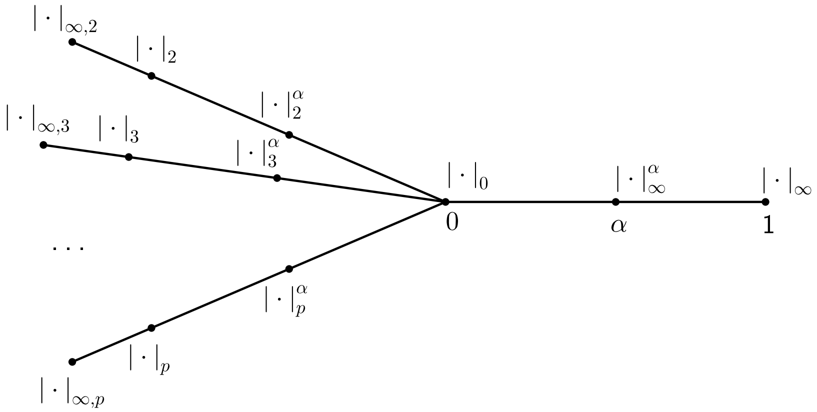

Figure 1: $\mathcal{M}(\mathbb{Z})$, the Berkovich spectrum of $\mathbb{Z}$.

10.3 What does $\mathfrak{N}$ look like?

The main tool is to cover it with the !-cover $\mathbb{Z}[\hat{q}^{\pm 1}]_{\text{gas}} \times (0,1) \to \mathfrak{N}$ we just constructed.

Theorem 10.16. *The morphism*

$$\prod_{n\in\mathbb{N}} N(n) : \mathfrak{N} \to \prod_{n\in\mathbb{N}} [0,\infty]$$

surjects onto the extended *Berkovich spectrum of* $\mathbb{Z}$*.*

Definition 10.17 (Berkovich spectrum).

- A Banach ring is a commutative ring with unit A together with a map $|\cdot|_A : A \to \mathbb{R}$ such that
  1. $|0|_A = 0$ and $|\pm 1|_A = 1$ (or $A = 0$) ;
  2. $|x+y|_A \leq |x|_A + |y|_A$;
  3. $|xy|_A \leq |x|_A \cdot |y|_A$;
  4. A is complete with respect to $|\cdot|_A$.
- If A is a Banach ring, we define its Berkovich spectrum as

$$\mathcal{M}(A) \coloneqq \left\{ |\cdot| : A \to \mathbb{R}_{\geq 0} \text{ s.t. } \begin{array}{r} \text{a) } |\cdot| \leq |\cdot|_A \\ \text{b) 1 and 2 hold} \\ \text{c) } |x\cdot y| = |x||y| \end{array} \right\} \subseteq \prod_{f\in A} [0,\infty]$$

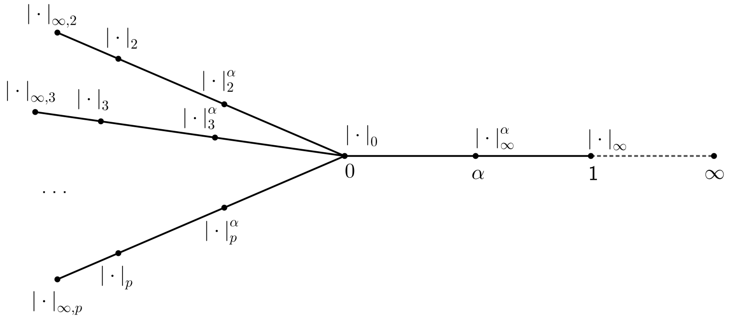

Figure 2: $\mathcal{M}(\mathbb{Z})^{\text{ext}}$, the extended Berkovich spectrum of $\mathbb{Z}$.

endowed with the subspace topology.

For $\mathbb{Z}$ endowed with its usual absolute value norm $|\cdot|_\infty$, the elements of the Berkovich spectrum are the following.

- the p-adic branch: for every prime p, we have the pullback of the trivial norm on $\mathbb{F}_p$ along $\mathbb{Z} \to \mathbb{F}_p$, we call it $|\cdot|_{\infty,p}$. Moreover, we have the p-adic norm $|\cdot|_p$ and its powers $|\cdot|_p^\alpha$ for all $\alpha \in (0,\infty)$.
- the archimedean branch: we have the archimedean norm $|\cdot|_\infty$ and its powers $|\cdot|_\infty^\alpha$, with $\alpha \in (0,1]$.
- the central vertex, corresponding to the trivial norm $|\cdot|_0$.

Theorem 10.16 states that $\mathfrak{N} \to \prod_{n\in\mathbb{Z}}[0,\infty]$ almost factors through $\mathcal{M}(\mathbb{Z})$: indeed it factors over $\mathcal{M}(\mathbb{Z})^{\text{ext}}$, the extended Berkovich spectrum. This object is obtained from $\mathcal{M}(\mathbb{Z})$ by adding the segment $(1,\infty]$, which corresponds to the powers $|\cdot|_\infty^\alpha$ with $\alpha \in (1,\infty)$ and a point corresponding to $\alpha = \infty$. This can be represented as in Figure 2.

The dotted part of the image corresponds to archimedean "norms" where the triangular inequality fails.

Proof sketch. Pick a prime p and consider $N(p) : \mathfrak{N} \to [0,\infty]$.

- Let us study the locus where $N(p) \subseteq (0,1)$ (i.e. $N(p)^{-1}((0,1))$). By Theorem 10.15, for every analytic ring A we have

  $$\big\{(N,p) \text{ s.t. } N(p) \subseteq (0,1)\big\} \simeq \{\text{maps AnSpec}(A) \to \text{AnSpec}(\mathbb{Z}[\hat{q}^{\pm 1}]_{\text{gas}}/(q-p))\times(0,1)\big\}.$$

Since their functor of points coincide on affine analytic stacks, the locus where $N(p) \subseteq (0,1)$ is the analytic stack

$$\mathrm{AnSpec}(\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}}/(q-p)) \times (0,1) = \mathrm{AnSpec}(\mathbb{Q}_{p,\mathrm{gas}}) \times (0,1).$$

- Let us study now the locus where $N(p) \subseteq (1,\infty)$ (i.e. $N(p)^{-1}((1,\infty))$). Again by Theorem 10.15, this locus is the analytic stack

$$\mathrm{AnSpec}(\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}}/(pq-1)) \times (0,1) = \mathrm{AnSpec}(\mathbb{R}_{\mathrm{gas}}) \times (0,1).$$

Remark 10.18. In particular, the preimage of the non-archimedean norm $|-|_\infty$ corresponds to $\frac{1}{p} \in \mathrm{AnSpec}(\mathbb{R}_{\mathrm{gas}}) \times (0,1)$, because $|-|_\infty$ is such that $|\frac{1}{p}|_\infty = \frac{1}{p}$.

We still need to analyse the loci where $N(p) = 0$, $N(p) = 1$ and $N(p) = \infty$. Apriori, we should study also the loci relative to other primes q, but actually the norm of p determines the norm of the other primes in almost all the cases. Indeed, we have the following

Lemma 10.19 (Triangle inequality)**.** *Let A be a normed analytic ring and let p be a prime.*

1. *If $N(p) \subseteq [0,1]$, then we have "$N(x+y) \leq \max\{N(x), N(y)\}$".*
2. *If $N(p) \subseteq [0,p]$, then we have "$N(x+y) \leq N(x) + N(y)$".*
3. *If $N(p) \subseteq [0,\infty)$, then there is a constant c such that "$N(x+y) \leq c \cdot (N(x) + N(y))$".*

Idea of proof. To prove this, we should use Lemma 10.14 and reduce everything to computations in $\mathrm{AnSpec}(\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}})$. □

In particular, this implies the following facts:

- by part 3, if there exists a prime for which $N(p) = \infty$, then we have $N(l) = \infty$ for every other prime l. Every norm satisfying this property is mapped to the limit point of the archimedean branch in $\mathcal{M}(\mathbb{Z})^{\mathrm{ext}}$.
- by part 1, there exists at most one prime p for which we have $N(p) = 0$ (otherwise, writing 1 as a linear combination of such two different primes p and l, we would get $N(1) = 0$). Any such norm is mapped to the limit point of the p-adic branch.
- by part 1, if $N(p) \subseteq (0,1)$, then $N(l) = 1$ for all $l \neq p$ and such norms are mapped to the interior of the p-adic branch;
- by parts 2 and 3, if $N(p) \subseteq (1,\infty)$ for some p, then such norms are mapped to the interior of the extended archimedean branch (the classical one if $N(p) \subseteq (1,p]$, the dotted one if $N(p) \subseteq (p,\infty)$).

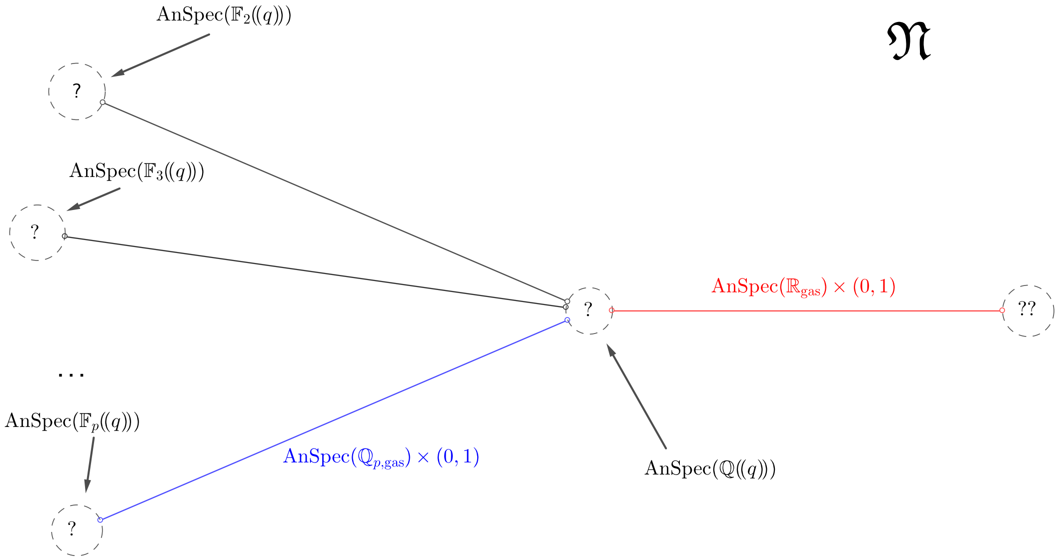

Figure 3: $\mathfrak{N}$, the stack of norms.

This shows that the image of $\mathfrak{N} \to \prod_{n\in\mathbb{Z}}[0,\infty]$ is contained in $\mathcal{M}(\mathbb{Z})^{\text{ext}}$. To show the equality, the only thing left to show is that the limit points have a preimage:

- there is a norm on the analytic ring $\mathbb{Q}((q))$ such that $N(p) = 1$ for all primes p. Thus AnSpec($\mathbb{Q}((q))$) maps to the preimage of the central vertex.
- for all prime p, there is a norm on the analytic ring $\mathbb{F}_p((q))$ such that $N(p) = 0$ and $N(l) = 1$ for all $l \neq p$. Thus AnSpec($\mathbb{F}_p((q))$) maps to the preimage of the limit point of the p-adic branch.
- there is an analytic ring together with a norm N such that $N(p) = \infty$ for all prime p (quite mysterious). This analytic ring maps to the preimage of the limit point of the archimedean branch.

$\square$

We have the following description of $\mathfrak{N}$. It is an analytic stack which lives over $\mathcal{M}(\mathbb{Z})^{\text{ext}}$, and we can describe (almost) all fibers. The preimage of any open branch is quite simple to describe, as it really looks like the space $(0,1)$. However, the preimage of the central vertex and of the limit points are quite unclear. We can say that the preimage of the central vertex receives a map from AnSpec($\mathbb{Q}((q))$) and that the preimage of the limit point of the p-adic branch receives a map from AnSpec($\mathbb{F}_p((q))$).

10.4 More from the discussion session

During the discussion session, some topics from this talk have been addressed in more detail. We present the most relevant comments in this section.

10.4.1 More on condition 4

Let A be an analytic ring and let $N : \mathbb{P}^1_A \to [0,\infty]$ be a norm. By condition 4 of Definition 10.2 we have a factorisation

$$\begin{tikzcd}
\mathrm{AnSpec}(A[\hat{T}]) = \mathbb{D}_A \arrow[r, "\varphi"] \arrow[rrd, dashed] & \mathbb{P}^1_A \arrow[r, "N"] & {[0,\infty]} \\
 & & {[0,1]} \arrow[u, hook]
\end{tikzcd}$$

Moreover, the map φ is an isomorphism over $[0,1)$. However, it is *not* an immerson.

Intuitively, we have

$$A[\hat{T}] = \{\sum_{n=0}^{\infty} a_n T^n \,|\, \text{for all } M \in \mathrm{Mod}_A \text{ and all } (m_n) \subseteq M \text{ null-sequence, we have } \sum_{n=0}^{\infty} a_n m_n \in M\}.$$

We have maps

$$\mathcal{O}(\{|T| \leq 1\})^\dagger \to A[\hat{T}] \to \mathcal{O}(\{|T| < 1\}).$$

The fact that φ becomes an isomorphism over $[0,1)$ means that we have

$$A[\hat{T}] \otimes^{\mathrm{L}}_{A[T]} \mathcal{O}(\{|T| < r\})^\dagger = \mathcal{O}(\{|T| \leq r\})^\dagger \qquad \forall r < 1.$$

As an example, take $A = \mathbb{Q}_p$ and consider the norm defined by overconvergent Tate algebras. In this case we have

$$A[\hat{T}] = \mathbb{Z}_p[\![T]\!]\left[\frac{1}{p}\right],$$

i.e. bounded functions in a unit disk.

10.4.2 Analytic Berkovich spectra

Let $(A, |\cdot|_A)$ be a Banach ring. We define an analytic stack

$$\mathrm{AnSpec}^{\mathrm{Berk}}(A) \subseteq \mathrm{AnSpec}(A) \times \mathfrak{N}$$

which is called *analytic Berkovich spectrum* of $(A, |\cdot|_A)$. To do it, we first observe that a map $\mathrm{AnSpec}(B) \to \mathrm{AnSpec}(A) \times \mathfrak{N}$ is given by a couple (φ, N) where $\varphi : A \to B$ is a morphism of analytic rings and N is a norm on B. Once we have this, we can define a map

$$\prod_{f \in A} N(\varphi(f)) : \mathrm{AnSpec}(B) \to \prod_{f \in A} [0,\infty].$$

To make the notation lighter, we denote this map just by $\prod_{f\in A} N(f)$. Now we define $\mathrm{AnSpec}^{\mathrm{Berk}}(A)$ as follows. For every analytic ring B, we set

$$\mathrm{Hom}_{\mathrm{AnStack}}(\mathrm{AnSpec}(B), \mathrm{AnSpec}^{\mathrm{Berk}}(A)) \coloneqq \left\{ \begin{array}{c} \text{maps } \mathrm{AnSpec}(B) \to \mathrm{AnSpec}(A) \times \mathfrak{N} \\ \text{s.t. } \prod_{f\in A} N(f) \text{ factors through } \mathcal{M}(A)_{\mathrm{Betti}} \end{array} \right\}$$

Remark 10.20. The condition that $\prod_{f\in A} N(f)$ factors through $\mathcal{M}(A)$ ensures that we have $N(f) \subseteq [0, |f|_A]$ for all $f \in A$.

One can prove that this defines an analytic stack $\mathrm{AnSpec}^{\mathrm{Berk}}(A)$ which has a canonical map

$$\mathrm{AnSpec}^{\mathrm{Berk}}(A) \to \mathcal{M}(A)_{\mathrm{Betti}}.$$

One can globalise this construction. One can also see that if X is a complex analytic variety, the map

$$X \to X(\mathbb{C})_{\mathrm{Betti}}$$

presented in Talk 8 fits in this framework. Indeed, one can show that $X(\mathbb{C})$ coincides with the Berkovich spectrum $\mathcal{M}(X)$ by using Gelfand-Mazur theorem.

10.4.3 More examples of open and closed immersions

These examples show the flexibility of working with analytic stacks, where we are able to detect a lot of open and closed subspaces.

Example 10.21. If R is a finite type $\mathbb{Z}$-algebra and $f \in R$, we have morphisms

$$R_{\square} \to (R\left[\frac{1}{f}\right], R)_{\square} \to R\left[\frac{1}{f}\right]_{\square},$$

where the first one is an idempotent localisation.

- $\mathrm{AnSpec}((R\left[\frac{1}{f}\right], R)_{\square})$ is *closed* in $\mathrm{AnSpec}(R_{\square})$. Its *open* complement is an infinitesimal open neighbourhood of $f = 0$, i.e. the (non-affine) analytic stack
$$\operatorname*{colim}_{n\in\mathbb{N}} \mathrm{AnSpec}((R/^L f^n, R)_{\square}).$$
This is an analytic stack whose quasi-coherent sheaves are f-adically complete R-modules.

- $\mathrm{AnSpec}(R\left[\frac{1}{f}\right]_{\square})$ is *open* in $\mathrm{AnSpec}(R_{\square})$ and it is a bit smaller than $\mathrm{AnSpec}((R\left[\frac{1}{f}\right], R)_{\square})$. Its *closed* compleement is the (affine) analytic stack
$$\mathrm{AnSpec}((R^{\wedge f}, R)_{\square}).$$
This is an analytic stack whose quasi-coherent sheaves are modules over the f-adic completion of R.

Remark 10.22. In the context of the previous remark,

$$\mathrm{AnSpec}((R^{\wedge f}, R)_\square) \sqcup \mathrm{AnSpec}((R\left[\frac{1}{f}\right], R)_\square)$$

forms a !-cover of $\mathrm{AnSpec}(R_\square)$ (use 5.12). Since we have !-descent of quasi-coherent sheaves, this recovers (a form of) the Beauville-Laszlo theorem.

Example 10.23. We already saw that an example of open immersion is

$$j : \mathrm{AnSpec}(\mathbb{Z}[T]_\square) \to \mathrm{AnSpec}((\mathbb{Z}[T], \mathbb{Z})_\square),$$

where the complementary closed is associated to the idempotent algebra $\mathbb{Z}((T^{-1}))$. To visualise it, it is better to base change this morphism to $\mathbb{Q}_{p,\square}$. Then we get the open immersion

$$\mathbb{D} \to \mathbb{A}^{1,\mathrm{alg}}_{\mathbb{Q}_{p,\square}},$$

where $\mathbb{D}$ is a closed affinoid disk. Here the complementary open is the analytic stack

$$\mathrm{AnSpec}(\mathbb{Z}_p((T^{-1}))\left[\frac{1}{p}\right]),$$

which intuitively is an open disk of radius 1 at ∞.

Since we constructed norms, we can give more examples of open immersions. If we endow $\mathbb{Q}_{p,\square}$ with the norm given by overconvergent Tate algebras, we can define $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{Q}_{p,\square}}$ as $N^{-1}([0,\infty))$. Thus we have the following commutative diagram

$$\begin{tikzcd}
 & \mathbb{A}^{1,\mathrm{an}}_{\mathbb{Q}_{p,\square}} \arrow[dl, "\text{open}"'] & \\
\mathbb{A}^{1,\mathrm{alg}}_{\mathbb{Q}_{p,\square}} & & \mathbb{D}. \arrow[ul] \arrow[ll, "\text{open}"']
\end{tikzcd}$$

The map $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{Q}_{p,\square}} \to \mathbb{A}^{1,\mathrm{alg}}_{\mathbb{Q}_{p,\square}}$ is an open immersion, where the complementary open is the affine analytic stack

$$\mathrm{AnSpec}(\{\text{ germs of meromorphic functions at } \infty\,\}).$$

10.4.4 The gaseous base stack

One can show that $\mathbb{R}_{>0,\mathrm{Betti}}$ acts on $\mathfrak{N}$. The quotient by this action contracts the "open branches" to points. These open points of $\mathfrak{N}/\mathbb{R}_{>0}$ are $\mathrm{AnSpec}(\mathbb{Q}_{p,\mathrm{gas}})$ for each p and $\mathrm{AnSpec}(\mathbb{R}_{\mathrm{gas}})$.

Definition 10.24. The *gaseous base stack* is the quotient $\mathfrak{N}/\mathbb{R}_{>0}$.

The map $\mathfrak{N} \to \mathcal{M}(\mathbb{Z})^{\mathrm{ext}}$ induces a map $\mathfrak{N}/\mathbb{R}_{>0} \to \mathcal{M}(\mathbb{Z})^{\mathrm{ext}}/\mathbb{R}_{>0}$.

11 Infinite-dimensional Arakelov geometry - by François Charles

11.1 Warm up: analytic pairs and coherent sheaves

Definition 11.1. An analytic pair (X, K) is the datum of a complex space and a compact K in X.

Remark 11.2. Intuition X should be thought of as a generic fiber and K as the points with good reduction in $X(\mathbb{C})$.

If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, we get a morphism

$$H^0(X, \mathcal{F}) \to H^0(K, \mathcal{F})$$

where $H^0(X, \mathcal{F})$ is a nuclear Fréchet space and we define $H^0(K, \mathcal{F})$ to be $\operatorname{colim}_{K \subset U} H^0(U, K)$ which is the dual of a nuclear Fréchet space.

Definition 11.3. Let (X, K) be an analytic pair and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. We define $H^0(X, K, \mathcal{F})$ to be the datum of $H^0(X, \mathcal{F})$ endowed with the bornology coming from $H^0(K, \mathcal{F})$. These elements form a category with well defined kernels, cokernels and strict maps.

Definition 11.4. A morphism of schemes $f : X \to Y$ is said to be a strict map if $\operatorname{coker} f = \operatorname{coim} f$.

Definition 11.5. The category described in Definition 11.3 is not abelian but quasi-abelian.

Remark 11.6. Let us do a few recollections.

1. **Open mapping theorem**: For Fréchet spaces or dual of Fréchet spaces, morphisms with finite cokernels are strict.
2. Stein spaces and their modifications have cohomological definitions.

Fact 11.7. A complex space is Stein if one of the following equivalent conditions holds:

1. For all coherent $\mathcal{O}_X$-modules $\mathcal{F}$ the equality

   $$\forall i > 0, \ H^i(X, \mathcal{F}) = 0$$

   holds.
2. For all epimorphism of coherent $\mathcal{O}_X$-modules $\mathcal{F} \twoheadrightarrow \mathcal{G}$, the morphism $H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G})$ is a (strict) surjection.

Definition 11.8. A *modification* $\nu : X \to Y$ i s a holomorphic map such that:

1. the map ν is proper,

2. there exists a finite subset $F \subset Y$ such that ν induces an isomorphism $\nu : X \smallsetminus \nu^{-1}(F) \xrightarrow{\sim} Y \smallsetminus F$,
3. the equality $\nu_* \mathcal{O}_X = \mathcal{O}_Y$ holds.

Fact 11.9. Let X be a connected complex space. The following are equivalent:

1. the space X is a modification of a Stein space.
2. For all coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and for all positive integers i, the group $H^i(X, \mathcal{F})$ is finite-dimensional.
3. All surjections of coherent $\mathcal{O}_X$-modules $\mathcal{F} \twoheadrightarrow \mathcal{G}$ are (strict) with finite-dimensional cokernel.

Remark 11.10. Strictness of morphisms is somewhat subtle.

Example 11.11. Let $\mathcal{F} \hookrightarrow \mathcal{G}$ and $K \hookrightarrow X$. Then the morphism $H^0(K, \mathcal{F}) \hookrightarrow H^0(K, \mathcal{G})$ is strict.

Example 11.12. Let $\mathcal{F} \twoheadrightarrow \mathcal{G}$ and $K \hookrightarrow X$. The morphism $H^0(K, \mathcal{F}) \to H^0(K, \mathcal{G})$ is not strict in general.

Example 11.13. Let $\varepsilon > 0$. We set $X = \mathbb{C}^2$ and $K = \{(z_1, z_2) \in D^2, |z_1| \leqslant \varepsilon \text{ or } |z_2| \geqslant 1 - \varepsilon\}$. Any holomorphic function on K extends to $\overline{D^2}$. Hence the morphism

$$H^0(\overline{D^2}, \mathcal{O}_{\mathbb{C}^2}) = H^0(K, \mathcal{O}_{\mathbb{C}^2}) \to H^0(K, \mathcal{O}_{\mathbb{C} \times \{0\}})$$

has dense image and is not strict.

Definition 11.14. Let X be Stein, $K \hookrightarrow X$ compact. We define $H^0(X, K, \mathcal{F})$ to be the datum of $H^0(X, \mathcal{F})$ endowed with bornology coming from $H^0(K, \mathcal{F})$.

Definition 11.15. We keep notation of Definition 11.14. We define

$$\widehat{K} := \{x \in X \mid \forall f \in \mathcal{O}_X(X) \text{ such that } \|f\|_K^\infty \leqslant 1, \ |f(x)| \leqslant 1\}$$

the holomorphic convex hull of K in X.
If K equals $\widehat{K}$, we say that K is holomorphically convex in X.

Exercise 11.16. Let us keep notation of Definition 11.14. Let x in X. The morphism

$$H^0(X, K, \mathcal{O}_X) \to H^0(X, K, \mathcal{O}_{\{x\}})$$

is strict if and only if x is in K or in $X \smallsetminus \widehat{K}$.

Definition 11.17. Let (X, K) be an analytic pair. We say that (X, K) is Stein (respectively mod-Stein) if for all epimorphisms of coherent $\mathcal{O}_X$-modules $\mathcal{F} \twoheadrightarrow \mathcal{G}$, the morphism $H^0(X, K, \mathcal{F}) \to H^0(X, K, \mathcal{G})$ is a strict surjection (respectively strict with finite-dimensional cokernel).

Theorem 11.18 (Structure theorem). *1. An analytic pair (X, K) is Stein if and only if X is Stein and K is holomorphically convex in X.*

2. *An analytic pair (X, K) is mod-Stein if and only if X is mod-Stein, the compact K contains all the positive dimensional subvarieties of X and $K = \widehat{K}$.*

Example 11.19. Let X be proper and $\overline{L}$ be a Hermitian bundle on X. We may look at

$$\mathbb{V}(\overline{L}) = (\mathrm{Spec}(\mathrm{Sym}^\bullet L), K = \{x, \varphi \in L_x^\vee, \|\varphi\| \leqslant 1\})$$

where $\mathrm{Spec}(\mathrm{Sym}^\bullet L)$ is the total space of $L^\vee$.

Theorem 11.20. *We keep notation of Example 11.19. The Hermitian bundle $\overline{L}$ is ample if and only if $\mathbb{V}(\overline{L})$ is mod-Stein.*

11.2 Back to arithmetic

Recall 11.21. An A-scheme is a pair (X, K) with X a separated scheme of finite type over $\mathbb{Z}$ and $K \subset X(\mathbb{C})$ is compact and invariant by complex conjugation.

Remark 11.22. Relative arguments in A-schemes work well by mixing algebraic geometry and arguments on analytic pairs.

Definition 11.23. We may define relatively affine (respectively mod-affine) A-schemes by X is affine (respectively mod-affine) and (X^{an}, K) is Stein (respectively mod-Stein).

Definition 11.24. Given a Hermitian line bundle $\overline{L}$ on X proper, we may define $\mathbb{V}(\overline{L})$.

Definition 11.25. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let us define the notion of a Hermitian coherent sheaf as a pair

$$\mathbb{V}(\mathcal{F}) = (\mathrm{Spec}(\mathrm{Sym}^\bullet F), T)$$

where T is compact and $\mathbb{C}^\times$-invariant.

Definition 11.26. Let us consider $(\mathrm{Spec}(\mathbb{Z}), \{*\})$, an A-scheme (X, K) and $\mathcal{F}$ coherent. We define $H^0(X, K, \mathcal{F})$ to be the datum of $H^0(X, \mathcal{F})$ endowed with the bornology coming from the countable $\mathbb{Z}$-module $H^0(K, \mathcal{F})$.

Remark 11.27. To study global aspects of A-schemes, we want to study "geometry of numbers" for these lattices and the bornology. There are two issues.

1. We work with bornology instead of norms.

   Example 11.28. Considering $\mathbb{V}(\overline{L})$, the equality $H^0(\mathbb{V}(\overline{L}), \mathcal{O}) = \bigoplus_{d \geqslant 0} H^0(X, L^{\otimes d})$ holds.

2. We need infinite-dimensional theory.

11.3 θ-invariants

Let us consider $\overline{E} = (E = \mathbb{Z}^n, \|\cdot\|)$ a euclidean lattice.

Definition 11.29. We define

$$h^0_\theta(\overline{E}) = \log\left(\sum_{v\in E} e^{-\pi\|v\|^2}\right) \quad \text{and} \quad h^1_\theta(\overline{E}) = h^0_\theta(\overline{E}^\vee)$$

which are non-negative real numbers.

Remark 11.30. These numbers should be thought of as a dimension verifying the Serre duality, even though we have not defined any H^0_θ or H^1_θ.

Definition/Proposition 11.31 (Poisson formula)**.** *The equality*

$$h^0_\theta(\overline{E}) - h^1_\theta(\overline{E}) = \deg \overline{E} := -\log(\operatorname{vol}(E_\mathbb{R}/E))$$

holds.

Proposition 11.32 (Functoriality)**.** *Let us suppose that we have a short exact sequence of euclidean lattices*

$$0 \to \overline{E} \to \overline{F} \to \overline{G} \to 0$$

id est it is a short exact sequence of abelian groups which is orthogonal for the metric. Then the inequalities

$$h^0_\theta(\overline{E}) + h^0_\theta(\overline{G}) - h^1_\theta(\overline{E}) \leqslant h^0_\theta(\overline{F}) \leqslant h^0_\theta(\overline{E}) + h^0_\theta(\overline{G})$$

hold.

Definition 11.33. Consider a morphism of euclidean lattices $\overline{E} \xrightarrow{\varphi} \overline{F}$. Let us define

$$\operatorname{rk}^1_\theta(\varphi) = h^1_\theta(\overline{F}) - h^1_\theta(\overline{F/\varphi(E)})$$

which we call the *rank* of φ.

Remark 11.34. The rank of a morphism $\varphi : \overline{E} \to \overline{F}$ should be thought of as sort of $\operatorname{rk}(H^1_\theta(\varphi))$ even though we have not defined any $H^1_\theta(\varphi)$.

Proposition 11.35. *Let φ and ψ be composable morphisms of euclidean lattices. The following inequality*

$$\operatorname{rk}^1_\theta(\varphi\circ\psi) \leqslant \min(\operatorname{rk}^1_\theta(\varphi), \operatorname{rk}^1_\theta(\psi))$$

holds.

Now, we want to understand how it extends to the general setting.

Proposition 11.36. *In the infinite-dimensional setting:*

1. *The number h^0_θ extends in $[0, \infty]$.*

2. *The case of h^1_θ is more complicated, we get invariants by approximation (by quotients and subscheaves).*

Remark 11.37. To get theorems from geometry of numbers, we replace the rank by the trace. Start with $(E, \|\cdot\|) = \overline{E}$. θ-invariants of $\overline{E}$ control the geometry of $\overline{E}' := (E, \|\cdot\|')$ where $\|\cdot\|' \leqslant \|\cdot\|$ in terms of $Tr\frac{\|\cdot\|'^2}{\|\cdot\|^2}$.

Example 11.38 (Minkowski's first theorem)**.** We consider $\overline{E}$, $\overline{E}'$ and $\delta > 0$. Let us denote $\inf\{\|v\|, v \neq 0\}$ by $\lambda_1(E, \|\cdot\|)$.
The contrapositive of Minkowski's first theorem states:

$$\lambda_1(E, \|\cdot\|) \geqslant 1 \Rightarrow h^0_\theta(E, e^\delta\|\cdot\|) \leqslant \log\underbrace{\left(1 - \frac{1}{2}\left(Tr\frac{\|\cdot\|'^2}{\|\cdot\|^2}\right)e^{-2\delta}\right)}_{>0}.$$

A variant is the following statement: the number h^1_θ controls the covering radius.

11.4 Back to diophantine geometry

Let us consider (X, K) an A-scheme relatively (mod) affine.

Definition 11.39. We say that (X, K) is mod-affine if $h^1_\theta(X, K, \mathcal{F}) < \infty$ for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$.

Theorem 11.40. *Let (X, K) be mod-affine. Then, there exists a subscheme $Z \hookrightarrow X$ such that:*

1. *the scheme Z is proper over $\mathbb{Z}$ and purely positive dimensional.*
2. *$Z(\mathbb{C}) \hookrightarrow K$ is maximal.*

Remark 11.41. All h^1_θ "come from Z".

Remark 11.42. We keep notation of Theorem 11.40. Concretely, if X is affine:

1. there are finitely many integral points in $X(\mathbb{C})$.
2. these points are the only obstruction to approximating relative functions in K by $\mathcal{O}_X(X)$.

Example 11.43. Let X be proper. A Hermitian line bundle $\overline{L}$ on X proper is ample if and only if $\mathbb{V}(\overline{L})$ is mod-affine.

Example 11.44. Let X be proper. Let $D \hookrightarrow X$ be a divisor with an ample normal bundle

$$K \hookrightarrow (X \smallsetminus D)(\mathbb{C}).$$

Then, $(X \smallsetminus D, K)$ is mod-affine.

12 Final questions

At the end, some questions (without answer for the moment) were raised. Their goal is to understand the relationship between Arakelov geometry (and in particular the objects introduced in Talks 9 and 11) and the theory of analytic stacks. The idea is that objects from Arakelov geometry should be interpreted by means of analytic stacks living over the gaseous base stack $\mathfrak{N}/\mathbb{R}_{>0}$.

We recall that the gaseous base stack $\mathfrak{N}/\mathbb{R}_{>0}$ (see Definition 10.24) lives over $\mathcal{M}(\mathbb{Z})^{\text{ext}}/\mathbb{R}_{>0}$. We recall that the archimedean branch in $\mathcal{M}(\mathbb{Z})^{\text{ext}}$ consists in the open interval corresponding to archimedean "norms" and a mysterious limit point at ∞: in what follows, we call Z the image of this branch along $\mathcal{M}(\mathbb{Z})^{\text{ext}} \to \mathcal{M}(\mathbb{Z})^{\text{ext}}/\mathbb{R}_{>0}$. Hence in Z we have the "open" point (whose preimage in $\mathfrak{N}/\mathbb{R}_{>0}$ is the archimedean norm on $\mathrm{AnSpec}(\mathbb{R}_{\text{gas}})$) and a mysterious limit point. A similar description can be given for other branches, with a p-adic norm on $\mathrm{AnSpec}(\mathbb{Q}_{p,\text{gas}})$ taking the role of the archimedean norm on $\mathrm{AnSpec}(\mathbb{R}_{\text{gas}})$.

1. (Analytic pairs) Let X be a complex analytic manifold. Then the analytic stack X^{an} lives over the gaseous base stack $\mathfrak{N}/\mathbb{R}_{>0}$. Indeed, we have morphisms of analytic stacks

   $$X^{\text{an}} \to \mathrm{AnSpec}(\mathbb{C}_{\text{gas}}) \hookrightarrow \mathfrak{N}/\mathbb{R}_{>0},$$

   where the last morphism is an open immersion. The question is the following: what is the condition on X so that X^{an} has a model over $\mathfrak{N}/\mathbb{R}_{>0} \times_{\mathcal{M}(\mathbb{Z})^{\text{ext}}/\mathbb{R}_{>0}} Z$? More precisely, is there a relation between this condition and X having a Kähler structure?

   This guess is justified by the following two facts

   (a) when X is a complex proper smooth analytic variety, endowed with a Kähler structure, one has the degeneration of the Hodge to de Rham spectral sequence, provided by Hodge theory.

   (b) For an algebraic variety X over $\mathbb{Q}_p$ of dimension $d < p$, the same degeneration holds whenever there is a model $\mathcal{X}$ of X over $\mathbb{Z}_p$ by results of Deligne–Illusie.

   Having a model over

   $$\mathfrak{N}/\mathbb{R}_{>0} \times_{\mathcal{M}(\mathbb{Z})^{\text{ext}}/\mathbb{R}_{>0}} Z$$

   could give tools for proving the degeneration of the Hodge-de Rham spectral sequence, as an archimedean analog of Deligne–Illusie.

2. (Formal analytic arithmetic surface) Let V be an arithmetic surface and let $p : \mathrm{Spec}(\mathbb{Z}) \hookrightarrow V$ be an integral point. We saw that completing V along this integral point gives a formal-analytic arithmetic surface $(\mathcal{V}^{\wedge p}, K)$, where K is a compact Riemann surface.

Let now $\mathcal{V}$ be the analytic stack associated to V. By base-changing it to $\mathfrak{N}/\mathbb{R}_{>0}$ we obtain $\mathcal{V}\times\mathfrak{N}/\mathbb{R}_{>0}$, an analytic stack living over the gaseous base stack. Any integral point $p:\mathrm{Spec}(\mathbb{Z})\hookrightarrow V$ give rise to a section

$$p:\mathfrak{N}/\mathbb{R}_{>0}\to\mathcal{V}\times\mathfrak{N}/\mathbb{R}_{>0}.$$

The question is the following: how is $(\mathcal{V}^{\wedge p},K)$ related to a closed neighbourhood of p in $\mathcal{V}\times\mathfrak{N}/\mathbb{R}_{>0}$?

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