# The work of Drinfeld

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The goal of the talk was to justify the central role played by moduli spaces of shtukas in the Langlands program, by giving a brief overview of the work of Drinfeld on the global Langlands correspondence for function fields. This is a big and deep subject and we decided to focus on the results of [3] and on the relation between elliptic modules and shtukas with two legs. Our discussion follows closely [1] and [2].

As usual, let $k = \mathbf{F}_q$, $X$ a smooth projective, geometrically connected curve over $k$ and $F = k(X)$. Choose a point $\infty \in |X|$, and assume for simplicity that $\deg(\infty) = 1$. Let $F_\infty$ be the completion of $F$ at $\infty$, $\mathbf{C}_\infty$ be the completion of a separable closure $\overline{F}_\infty$ of $F_\infty$, and $A = H^0(X \setminus \{\infty\}, \mathcal{O})$.

## 1. Elliptic modules

1.1. **Definition.** The seed of shtukas were Drinfeld's *elliptic modules*. Let $\mathbf{G}_a$ be the additive group, and $K$ a characteristic $p$ field. We set $K\{\tau\} = K \otimes_{\mathbf{Z}} \mathbf{Z}[\tau]$, with multiplication given by

$$(a \otimes \tau^i)(b \otimes \tau^j) = ab^{p^i} \otimes \tau^{i+j}.$$

We have an isomorphism $K\{\tau\} \cong \mathrm{End}_K(\mathbf{G}_a)$ sending $\tau$ to $X \mapsto X^p$. If $a_m$ is the largest non-zero coefficient, then the *degree* of $\sum_{i=0}^m a_i\tau^i \in K\{\tau\}$ is defined to be $p^m$. The *derivative* is defined to be the constant term $a_0$.

*Definition* 1.1. Let $r > 0$ be an integer and $K$ a characteristic $p$ field. An *elliptic $A$-module of rank $r$* is a ring homomorphism

$$\phi\colon A \to K\{\tau\}$$

such that for all non-zero $a \in A$, $\deg \phi(a) = |a|_\infty^r$.

Let $S$ be a scheme of characteristic $p$. An *elliptic $A$-module of rank $r$ over $S$* is a $\mathbf{G}_a$-torsor $\mathcal{L}/S$, with a morphism of rings $\phi\colon A \to \mathrm{End}_S(\mathcal{L})$ such that for all points $s\colon \mathrm{Spec}\, K \to S$, the fiber $\mathcal{L}_s$ is an elliptic $A$-module of rank $r$.

*Remark* 1.2. The function $a \mapsto \phi(a)'$ (the latter meaning the derivative of $\phi(a)$) defines a morphism of rings $i\colon A \to \mathcal{O}_S$, i.e. a morphism $\theta\colon S \to \mathrm{Spec}\, A$.

1.2. **Level structures and moduli space.** Let $I$ be an ideal of $A$. Let $(\mathcal{L}, \phi)$ be an elliptic module over $S$. Assume for simplicity that $S$ is an $A[I^{-1}]$-scheme, i.e. the map $\theta$ factors through $\theta\colon S \to \mathrm{Spec}\, A \setminus V(I)$.

Let $\mathcal{L}_I$ be the group scheme defined by the equations $\phi(a)(x) = 0$ for all $a \in I$. This is an étale group scheme over $S$ with rank $\#(A/I)^r$. An $I$-level structure on $(\mathcal{L}, \varphi)$ is an $A$-linear isomorphism $\alpha\colon (I^{-1}/A)_S^r \xrightarrow{\sim} \mathcal{L}_I$.

Choose $0 \subsetneq I \subsetneq A$. We have a functor

$$F_I^r\colon A[I^{-1}] - \mathbf{Sch} \to \mathbf{Sets}$$

sending $S$ to the set of isomorphism classes of elliptic $A$-modules of rank $r$ with $I$-level structure, with $\theta$ being the structure morphism.

**Theorem 1.3** (Drinfeld)**.** *$F_I^r$ is representable by a smooth affine scheme $M_I^r$ over $A[I^{-1}]$.*

## 2. Analytic theory of elliptic modules

2.1. **Description in terms of lattices.** Let $\Gamma$ be an $A$-lattice in $\mathbf{C}_\infty$ (that is, a discrete additive subgroup of $\mathbf{C}_\infty$ which is an $A$-module.) Then we define

$$e_\Gamma(x) = x \prod_{x \in \Gamma - 0} (1 - x/\gamma).$$

Drinfeld proved that this is well-defined for all $x \in \mathbf{C}_\infty$, and induces an isomorphism of abelian groups $e_\Gamma \colon \mathbf{C}_\infty/\Gamma \xrightarrow{\sim} \mathbf{C}_\infty$. This allows to define a function $\phi_\Gamma \colon A \to \mathrm{End}_{\mathbf{C}_\infty}(\mathbf{G}_a)$, by transporting the $A$-module structure on the left-hand side to the right-hand side, which only depends on the homothety class of the $A$-lattice $\Gamma$.

The following theorem is reminiscent of the description of elliptic curves over $\mathbf{C}$.

**Theorem 2.1** (Drinfeld)**.** *The function $\Gamma \mapsto \phi^\Gamma$ induces a bijection between*

$$\left\{ \begin{array}{c} \textit{rank } r \textit{ projective } A\textit{-lattices} \\ \textit{in } \mathbf{C}_\infty/\textit{homothety} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{rank } r \textit{ elliptic } A\textit{-modules} \\ \textit{over } \mathbf{C}_\infty \textit{ such that } \phi(a)' = a \\ /\textit{isomorphism} \end{array} \right\}$$

*Remark* 2.2. Under this bijection, an $I$-level structure equivalent to an $A$-linear isomorphism $(A/I)^r \cong \Gamma/I\Gamma$ for the lattices.

2.2. **Uniformization.** We now try to parametrize the objects on the left hand side of (2.1). Let $Y$ be a projective $A$-module of rank $r$. Then we have a bijection

$$\left\{ \begin{array}{c} \text{homothety classes of } A\text{-lattices in } \mathbf{C}_\infty \\ \text{isomorphic to } Y \text{ as } A\text{-modules} \end{array} \right\} \leftrightarrow \mathbf{C}_\infty^\times \backslash \mathrm{Inj}(F_\infty \otimes_A Y, \mathbf{C}_\infty)/\operatorname{GL}_A(Y).$$

Next we observe that there is a bijection (after fixing an identification $F_\infty \otimes_A Y = F_\infty^r$)

$$\mathbf{C}_\infty^\times \backslash \mathrm{Inj}(F_\infty \otimes_A Y, \mathbf{C}_\infty) \leftrightarrow \mathbf{P}^{r-1}(\mathbf{C}_\infty) \setminus \bigcup (F_\infty\text{-rational hyperplanes}),$$

given by sending $u \in \mathrm{Inj}(F_\infty \otimes_A Y, \mathbf{C}_\infty)$ to $[u(e_1) : \ldots : u(e_r)]$ ($(e_1, \ldots, e_r)$ is the canonical basis of $F_\infty^r$). The right-hand side is the set of $\mathbf{C}_\infty$-points of the famous *Drinfeld upper half-space* $\Omega^r$.

As Spec $A = X \setminus \{\infty\}$, a projective $A$-module of rank $r$ is the same as a vector bundle of rank $r$ on $X \setminus \{\infty\}$. Using Weil's adélic description of vector bundles, one finally gets

$$M_I^r(\mathbf{C}_\infty) \cong \operatorname{GL}_r(F) \backslash (\Omega^r(\mathbf{C}_\infty) \times \operatorname{GL}_r(\mathbf{A}_F^\infty)/\operatorname{GL}_r(\widehat{A}, I)),$$

where $\operatorname{GL}_r(\widehat{A}, I) := \ker\left(\operatorname{GL}_r(\widehat{A}) := \prod_{v \neq \infty} \operatorname{GL}_r(\mathcal{O}_v) \to \operatorname{GL}_r(A/I)\right)$. This bijection can be upgraded into an isomorphism of rigid analytic spaces :

**Theorem 2.3** (Drinfeld)**.** *One has an isomorphism of rigid analytic spaces over $F_\infty$ :*

$$M_I^{r,\mathrm{an}} = \operatorname{GL}_r(F) \backslash (\Omega^r \times \operatorname{GL}_r(\mathbf{A}_F^\infty)/\operatorname{GL}_r(\widehat{A}, I)).$$

## 3. Cohomology of $M_I^2$ and global Langlands for $\mathrm{GL}_2$

3.1. **Cohomology of the Drinfeld upper half plane.** We then briefly outlined Drinfeld's proof of global Langlands for $\mathrm{GL}_2$ using the moduli space of elliptic modules. Set $r = 2$, and $\Omega := \Omega^2$. Then one has

$$\Omega(\mathbf{C}_\infty) = \mathbf{P}^1(\mathbf{C}_\infty) \backslash \mathbf{P}^1(F_\infty).$$

There is a map $\lambda$ from $\Omega(\mathbf{C}_\infty)$ to the Bruhat-Tits tree, sending $(z_0, z_1)$ to the homothety class of the norm on $F_\infty^2$ defined by

$$(a_0, a_1) \in F_\infty^2 \mapsto |a_0 z_0 + a_1 z_1|,$$

and one can think to $\Omega$ as being a tubular neighborhood of the Bruhat-Tits tree. Using $\lambda$, one gets a quite explicit description of the geometry of the rigid analytic space $\Omega$ and proves that there is a $GL_2(F_\infty)$-equivariant isomorphism :

$$H^1_{\text{ét}}(\Omega_{\mathbf{C}_\infty}, \overline{\mathbf{Q}}_\ell) = (\mathcal{C}^\infty(\mathbf{P}^1(F_\infty), \overline{\mathbf{Q}}_\ell)/\overline{\mathbf{Q}}_\ell)^* \cong \mathrm{St}^*.$$

3.2. **Cohomology of $M_I^2$.** Now we use the uniformization of $M_I^2$ (theorem 2.3). Rewriting it as follows :

$$M_I^{2,\mathrm{an}} = \Big(\Omega \times \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F)/\,\mathrm{GL}_2(\widehat{A}, I)\Big)\,/\,\mathrm{GL}_2(F_\infty).$$

and using the Hochschild-Serre spectral sequence, we deduce a $\mathrm{GL}_2(\mathbf{A}_F)\times\mathrm{Gal}(\overline{F}_\infty/F_\infty)$-equivariant isomorphism[1] :

$$H^1_{\text{ét},!}(M_I^2 \otimes_F \overline{F}, \overline{\mathbf{Q}}_\ell) \cong \mathrm{Hom}_{\mathrm{GL}_2(F_\infty)}(\mathrm{St}, \mathcal{C}_0^\infty(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F)/\,\mathrm{GL}_2(\widehat{A}, I))) \otimes \mathrm{sp},$$

where sp is a 2-dimensional representation of $\mathrm{Gal}(\overline{F}_\infty/F_\infty)$ corresponding to the Steinberg representation by local Langlands. Drinfeld shows that

$$\varinjlim_I H^1_{\text{ét},!}(M_I^2 \otimes_F \overline{F}, \overline{\mathbf{Q}}_\ell) = \bigoplus_\pi \pi^\infty \otimes \sigma(\pi)$$

where $\pi$ runs over cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A}_F)$ with $\pi_\infty \cong \mathrm{St}$. Here $\sigma(\pi)$ is a degree two $\mathrm{Gal}(\overline{F}/F)$-representation. Moreover, Drinfeld shows that at unramified places, $\pi_v$ and $\sigma(\pi_v)$ correspond to each other by local Langlands.

*Remark* 3.1. This result is still quite far from the global Langlands correspondence for $GL_2$ over $F$, but it nevertheless allows to construct the local Langlands correspondence for $\mathrm{GL}_2$ over $K$, a characteristic $p$ local field, as was explained during the talk, by combining this global construction with the decomposition of global $L$ and $\epsilon$-factors as products of local constants (which is known to hold in positive characteristic) and a trick of twisting by a sufficiently ramified character. See [2].

## 4. From elliptic modules to shtukas

The relation between elliptic modules and shtukas passes through an intermediate object called an *elliptic sheaf.*

*Definition* 4.1. An *elliptic sheaf of rank $r > 0$ with pole at $\infty$* is a diagram

$$\begin{array}{ccccccccc}
\cdots & \hookrightarrow & \mathcal{F}_{i-1} & \xhookrightarrow{j_i} & \mathcal{F}_i & \xhookrightarrow{j_{i+1}} & \mathcal{F}_{i+1} & \hookrightarrow & \cdots \\
 & \nearrow & & \nearrow_{t_i} & & \nearrow_{t_{i+1}} & & \nearrow & \\
\cdots & \hookrightarrow & {}^\tau\mathcal{F}_{i-1} & \xhookrightarrow{{}^\tau j_i} & {}^\tau\mathcal{F}_i & \xhookrightarrow{{}^\tau j_{i+1}} & {}^\tau\mathcal{F}_{i+1} & \hookrightarrow & \cdots
\end{array}$$

(here as usual $\tau^*\mathcal{F} = (\mathrm{Id}_X \times \mathrm{Frob}_S)^*\mathcal{F})$ with $\mathcal{F}_i$ vector bundles of rank $r$, such that $j$ and $t$ are $\mathcal{O}_{X\times S}$-linear maps satisfying

(1) $\mathcal{F}_{i+r} = \mathcal{F}_i(\infty)$ and $j_{i+r} \circ \ldots \circ j_{i+1}$ is the natural map $\mathcal{F}_i \hookrightarrow \mathcal{F}_i(\infty)$.
(2) $\mathcal{F}_i/j_i(\mathcal{F}_{i-1})$ is an invertible sheaf along $\Gamma_\infty$.
(3) For all $i$, $\mathcal{F}_i/t_i(\tau^*\mathcal{F}_{i-1}) =$ is an invertible sheaf along $\Gamma_z$ for some $z\colon S \to X \setminus \{\infty\}$ (independent of $i$).
(4) For all geometric points $\overline{s}$ of $S$, the Euler characteristic $\chi(\mathcal{F}_0|_{X_{\overline{s}}})$ vanishes.

If $I$ is a non-zero ideal of $A$, there is also a natural notion of $I$-level structure on an elliptic sheaf over $S$, at least if $S$ lives over Spec $A \setminus V(I)$, and Drinfeld proves the following remarkable result.

**Theorem 4.2.** *Let $z\colon S \to$ Spec $A \setminus V(I)$. Then there exists a bijection, functorial in $S$, between the two sets :*

$$\left\{\begin{array}{c}\text{rank } r \text{ elliptic } A\text{-modules over } S\\ \text{with } I\text{-level structure}\\ \text{such that } \phi(a)' = z(a)\end{array}\right\}/\simeq\ \leftrightarrow \left\{\begin{array}{c}\text{rank } r \text{ elliptic sheaves over } S\\ \text{with } I\text{-level structure}\\ \text{and zero } z\end{array}\right\}/\simeq$$

[1]This is cheating a little : one has to apply carefully the Hochschild-Serre spectral sequence and one needs to introduce a compactification of $M_I^2$ to define the *cuspidal* cohomology of $M_I^2$ showing up on the left (corresponding to the space of cuspidal functions on the right).

The dictionary is explained in detail in [6] (see in particular the enlightening example $r = 1$ and its relation with geometric class field theory discussed there).

One shows that if $(\mathcal{F}., t., j.)$ is an elliptic sheaf, then for all $i$,

$$t_i(\tau^* \mathcal{F}_{i-1}) = \mathcal{F}_i \cap t_{i+1}(\tau^* \mathcal{F}_i), \text{ viewed as subsheaves of } \mathcal{F}_{i+1}.$$

Hence, one can actually reconstruct the entire elliptic sheaf from the triangle

$$\begin{array}{ccc} \mathcal{F}_0 & \xhookrightarrow{j} & \mathcal{F}_1 \\ & \overset{t}{\nearrow} & \\ {}^{\tau}\mathcal{F}_0 & & \end{array}$$

which is just a shtuka with two legs (one being fixed at $\infty$) ! One can not go in the other direction – shtukas with two legs are more general than elliptic sheaves. There is no direct analogy anymore between shtukas with one pole at $\infty$ and one zero $z$ and elliptic curves (or abelian varieties) but the family of stalks at closed points of $X$ of such a vector bundle, with their Frobenius, behaves somehow like the family of $\varphi$-modules attached to the reduction mod $\ell$ of the $p$-divisible group of an abelian variety over a number field, when the prime $\ell$ varies (the choice of $\ell$ corresponding roughly to the choice of a closed point and the choice of $p$ corresponding to the choice of $z$). Shtukas with two legs are the right objects to consider to prove the full Langlands correspondence for $GL_r$ (for all $r$) over a function field, as demonstrated by [4], [5].

## References


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