AN INTRODUCTION TO THE p-ADIC LOCAL LANGLANDS CORRESPONDENCE

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LECTURE I : INTRODUCTION

p fixed prime

Let $G = GL_2$ (alg. group over $\mathbb{Q}$). For each compact open subgp $K_f \subset G(\mathbb{A}_f)$, consider the Riemann surface $Y(K_f) := G(\mathbb{Q}) \backslash G(\mathbb{A}) / \mathbb{C}^* \cdot K_f$

L = finite ext of $\mathbb{Q}_p$ (field of coeff).

If W is an irreducible algebraic L-rep of G, consider

$$H_W := \varinjlim_{K_f} \underbrace{H^1}_{\text{Betti}}(Y(K_f), \underbrace{W}_{\text{seen as a local system on } Y(K_f)})$$

$G(\mathbb{A}_f)$ acts on the right on the tower of modular curves, hence it acts on H_W.

If K_f is a c.o.s of $G(\mathbb{A}_f)$, $(H_W)^{K_f} = \underbrace{H^1(Y(K_f), W)}_{\text{fin dim}}$ (apply Hochschild-Serre; does not work anymore with char p coeff.!)

$\Rightarrow$ the action of $G(\mathbb{A}_f)$ is smooth & admissible.

The Riemann surface $Y(K_f)$ has an algebraic model defined over $\mathbb{Q}$, for each K_f.

So if we look instead at étale cohomology of $Y(K_f)_{\overline{\mathbb{Q}}}$, we get a space (call it H_W again): H_W with two commuting actions of $G(\mathbb{A}_f)$ (smooth adm act) & $\mathcal{G}_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

We will only be interested by what happens "at p". So define:

$$H_W(K^p) = \varinjlim_{K_p \subset G(\mathbb{Q}_p)} H^1(Y(K_p K^p), W).$$

K^p: fixed cos of $G(\mathbb{A}_f^p)$

This a rep of $G(\mathbb{Q}_p) \times \mathcal{G}_{\mathbb{Q}}$. What does it look like?

Let f be a modular form, cuspidal of wt $k \geq 2$, level N. Recall that a Galois rep $\rho : \mathcal{G}_{\mathbb{Q}} \to GL_2(L)$ is said to be "attached to f" if for all $\ell \nmid N$, ρ is unr at ℓ & $\mathrm{tr}(\rho(\mathrm{Frob}_\ell)) = a_\ell$, where $\sum a_n q^n$ is the q-dev of f.

let $W_k = \mathrm{Sym}^{k-2}(L^2)$, $H_k := H_{W_k}$.

Th (Eichler-Shimura, Langlands, Deligne, Carayol)
I.1 let $\rho : \mathcal{G}_{\mathbb{Q}} \to GL_2(L)$ abs irred C° rep. let $k \geq 2$.

(i) $\mathrm{Hom}_{\mathcal{G}_{\mathbb{Q}}}(\rho, H_k)$ is $\neq 0 \iff \rho$ is attached to a modular cusp form of wt k.

(ii) In this case, $\exists$ irred smooth rep $\pi_p(\rho)$ of $G(\mathbb{Q}_p)$ over L st:

$$\mathrm{Hom}_{\mathcal{G}_{\mathbb{Q}}}(\rho, H_k(K^p)) = \pi_p(\rho)^{\oplus m(K^p)}$$

for some $\neq 0$ integer $m(K^p)$, $\forall K^p$ small enough.

(iii) $\pi_p(\rho)$ depends only on $\rho_p := \rho|_{\mathcal{G}_{\mathbb{Q}_p}}$.

So we can call $\pi_p(\rho) = \pi_p(\rho_p)$.

(Rk, $\pi_p(\rho_p)$ is the p-component of the aut. rep attached to f (suitably twisted, not the unitary one), but we will not need this.)

Let ρ be as in the previous thm. What kind of info does $\pi_p(\rho_p)$ contain? Can we recover ρ_p from $\pi_p(\rho_p)$? The answer is no. To give a precise statement, we first need some reminder on p-adic HT.

Brief reminder on p-adic HT : (ref: Berger,)

ρ_p as before is a p-adic rep of the Galois group of the p-adic field $\mathbb{Q}_p$. These reps are much more complicated objects than ℓ-adic reps of $\mathcal{G}_{p\text{-adic field}}$, where $\ell \neq p$. Fontaine understood how to isolate some interesting reps, using period rings. (For simplicity, $L = \mathbb{Q}_p$)

Let C = completion of an alg closure of $\mathbb{Q}_p$.

$$\tilde{\mathbb{E}}^+ = \mathcal{O}_C^\flat = \varprojlim_{x \mapsto x^p} \mathcal{O}_C \simeq \varprojlim_{F} \mathcal{O}_C/p \quad \text{perfect local ring of char } p.$$

$$\tilde{\mathbb{A}}^+ = W(\tilde{\mathbb{E}}^+) , \quad \tilde{\mathbb{B}}^+ = \tilde{\mathbb{A}}^+[\tfrac{1}{p}] = \Big\{ \sum_{k \gg -\infty} p^k [x_k] , \ x_k \in \tilde{\mathbb{E}}^+ \Big\}$$

Map : $\theta : \tilde{\mathbb{B}}^+ \to C$, $\sum p^k [x_k] \mapsto \sum p^k x_k^{(0)}$.

B^+_{dR} is defined as the completion of $\tilde{\mathbb{B}}^+$ for the $\ker(\theta)$-adic top.

$$B^+_{dR} := \varprojlim_n \tilde{\mathbb{B}}^+/(\ker\theta)^n.$$

let $\varepsilon = (\varepsilon^{(0)}, \varepsilon^{(1)}, \dots)$ be a compatible system of p^{th}-roots of 1 $\in \tilde{\mathbb{E}}^+$ $(\varepsilon^{(0)} = 1,\ \varepsilon^{(1)} \neq 1)$

$1 - [\varepsilon] \in \tilde{\mathbb{B}}^+$ & $\theta(1 - [\varepsilon]) = 0$.

Hence the series $-\sum_{n=1}^{\infty} \frac{(1-[\varepsilon])^n}{n}$ converges in $\mathbb{B}^+_{dR}$ to an element t ("$2i\pi$ of Fontaine"). ("$t = \log([\varepsilon])$")

If $g \in \mathcal{G}_{\mathbb{Q}_p}$, $g(t) = \log([\varepsilon^{\chi(g)}]) = \chi(g)\, t$.

cycl. character.

$$B_{dR} := B^+_{dR}[\tfrac{1}{t}].$$

It's a field, B^+_{dR} is a complete dvr with unif. t.

filtration on B_{dR} : $\mathrm{Fil}^i(B_{dR}) = t^i B^+_{dR}$, $i \in \mathbb{Z}$.

+ Galois action st $B_{dR}^{\mathcal{G}_K} = K$, for any p-adic field K.

Hence if V is a p-adic rep, $D_{dR}(V) := (B_{dR} \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_K}$ diag. action

is a filtered K-vs. One always has

$\dim_K D_{dR}(V) \leq \dim_{\mathbb{Q}_p} V$ & say

Def I.2 V is a de Rham rep of $\mathcal{G}_K$ if $\dim_K D_{dR}(V) = \dim_{\mathbb{Q}_p}(V)$.

HT wts = opposites of the jumps of the filtration on $D_{dR}(V)$

Thm I.3 (Fontaine-Messing, Faltings)

let X/K proper smooth variety. $\forall i \geq 0$, $H^i_{et}(X_{\bar{K}}, \mathbb{Q}_p)$ is a p-adic de Rham rep of $\mathcal{G}_K$, & $D_{dR}(H^i_{et}(X_{\bar{K}}, \mathbb{Q}_p)) = H^i_{dR}(X/K)$

as filtered K-vs Hodge filtration

This gives many interesting examples of de Rham reps!

(In the other direction, one has the famous

Conj (Fontaine-Mazur) let E be a number field, ρ a p-adic rep of $\mathcal{G}_E$. Assume ρ is odd unr. a.e. & ~~de Rham at~~ with $\neq$ HT wts places above p. Then ρ "comes from geometry" (arises as a subqt of a suitable Tate twist of the étale coh of some proper smooth var $/E$) / is modular)

Unfortunately, one cannot recover V with its Galois action from $D_{dR}(V)$. To do so, one has to restrict to a smaller class of p-adic reps & introduce other

- B_{cris} comes with a Galois action ($B_{cris}^{\mathcal{G}_K} = K_0$), and a Frobenius φ injective σ-semilinear
~~B_{st}~~ & an injective map: $B_{cris} \otimes_{K_0} K \to B_{dR}$.

Define $D_{cris}(V) = (B_{cris} \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_K}$.

& crystalline reps as before.

$D_{cris}(V)$ is a finite dim'l K_0-vs with: (i) a bijective Frobenius φ σ-semilinear, (ii) an exhaustive and separated K-linear filtration on
$D_{cris}(V) \otimes_{K_0} K \hookrightarrow D_{dR}(V)$ ($\simeq$ if crystalline).
= it's a filtered φ-module.

So far, no relation between Frobenius & filtration.

Hodge polygon Newton polygon

say D (= filtered φ-module) is admissible if $\forall D' \subset D$ subobject
Hodge pol(D') lies below Newton pol(D') + equality for $D' = D$.

For any V, crystalline, $D_{cris}(V)$ is admissible.

- B_{st} : * Galois action st. $B_{st}^{\mathcal{G}_K} = K_0$.
$\cup$ B_{cris} * injective σ-semilinear Frobenius φ ext. the one on B_{cris}
* K_0-linear derivation $N: B_{st} \to B_{st}$, st $N\varphi = p\varphi N$.

st. $B_{cris} = B_{st}^{N=0}$ (hence N is B_{cris}-linear!)

* injective (~~internal~~) map $K \otimes_{K_0} B_{st} \to B_{dR}$.

B_{st} obtain by adding to B_{cris} an element $\log_{\mathbb{G}_\alpha}(T)$, $\alpha \in L$ (B_{st} does not depend on α, but to map $K \otimes B_{st}$ to B_{dR}, have to make a choice of $\log(p)$)

$D_{st}(V) := (B_{st} \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_K}$ is a filtered (φ, N)-module, which is admissible (same def as before, but beware that D' is required to be N-stable!) if V is semi-stable. $\boxed{\text{cryst} \subset \text{ss} \subset \text{de Rham}}$

Ex: The functor D_{cris} is an equivalence between the cat of crystalline characters of $\mathcal{G}_K$ & 1-dim'l admissible (φ, N)-modules over K. The crystalline char are Tate twists of the $\mathbb{Z}_p^\times$-valued unramified characters of $\mathcal{G}_K$.

In general one has the deep result:

<u>Th</u> I.4 (Colmez-Fontaine) The functors

$$D_{cris} : \left\{ \begin{matrix} \text{cryst rep} \\ \text{of } \mathcal{G}_K \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{admissible} \\ \text{filtered } \varphi\text{-mod over} \\ K \end{matrix} \right\}$$

$$D_{st} : \left\{ \begin{matrix} \text{ss reps} \\ \text{of } \mathcal{G}_K \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{admissible filtered} \\ (\varphi, N)\text{-modules over } K \end{matrix} \right\}$$

are equivalences of categories.

Hence, crystalline & ss-reps can be described entirely by ~~of~~ semi-linear objects! <u>Ex of cryst / ss rep</u> : geometry.

<u>Th I.5</u> (Berger) The de Rham reps of $\mathcal{G}_K$ are exactly the potentially semi-stable reps of $\mathcal{G}_K$ (rep which become semi-stable over a certain finite ext^n of K).

Hence, to give a de Rham rep of $\mathcal{G}_K$, suffices to give an admissible filtered $(\varphi, N, \mathcal{G}_{\mathbb{Q}_p})$-module, namely $D_{pst}(V) = \bigcup_{[K':K] < \infty} D_{st}(V_{|\mathcal{G}_{K'}})$.

finite dim'l $\widehat{\mathbb{Q}}_p^{un}$-vs, semi-linear bij Frob φ
linear N, $N\varphi = p\varphi N$
semi-linear action of $\mathcal{G}_K$ commuting with N, φ.

filtered = exhaustive decreasing separated filtration on $D_{dR}(V) = (D_{pst}(V) \otimes \overline{\mathbb{Q}}_p)^{\mathcal{G}_K}$

admissible: $\forall$ subobject, etc.

<u>V de Rham rep</u>. The adm filtered $(\varphi, N, \mathcal{G}_{\mathbb{Q}_p})$-module can be turned into a Weil-Deligne rep by linearizing the action of $\mathcal{G}_{\mathbb{Q}_p}$: $g \in W_{\mathbb{Q}_p}$ acts as $g \cdot \varphi^{-\deg(g)}$. (Recall that a WD-rep is (r, N), r ~~smooth~~ rep of $W_{\mathbb{Q}_p}$ <u>c° for the discrete top</u> + monodromy N as before) Call it <u>$WD(D_{pst}(V))$</u>

<u>LLC</u> : $(G = GL_2)$

$$\left\{ \begin{matrix} \text{(admissible) irred} \\ \text{smooth reps of} \\ GL_2(\mathbb{Q}_p) \end{matrix} \right\}_{/\sim} \xleftrightarrow{LL} \left\{ \begin{matrix} \text{Frob semi-simple 2-dim'l} \\ \text{WD-reps} \end{matrix} \right\}_{/\sim}.$$

<u>Rk</u>: let V de Rham. Then

V becomes ss over an abelian ext of $\mathbb{Q}_p$ $\Leftrightarrow$ $WD(D_{pst}(V))$ ~~(as rep of $W_{\mathbb{Q}_p}$)~~ is reducible $\Leftrightarrow$ $LL(WD(D_{pst}(V)))$ is not supercuspidal

After this long digression, we can finally answer our original question:

<u>Th</u> I.6 (Saito) We have:

$$\pi_p(\rho_p) = LL(WD(D_{pst}(\rho_p)))$$

In particular, $\pi_p(\rho_p)$ only depends on $D_{pst}(V)$ viewed as a $(\varphi, N, \mathcal{G}_{\mathbb{Q}_p})$ module.

Forget two things: the HT weights & the filtration (actually HT wts = − jumps of the filtat°). There is thus no chance to recover ρ_p from $\pi_p(\rho_p)$!

Breuil had the following amazing idea, which was the starting pt of the p-adic LL program.

To each admissible filtration on the $(\varphi, N, \mathcal{G}_{\mathbb{Q}_p})$-module $D_{pst}(\rho_p)$ should correspond a suitable <u>completion</u> of the locally alg rep:

$$\pi^{alg}(\rho_p) = \underbrace{(\mathrm{Sym}^{k-2}(L^2))^*}_{alg} \otimes \underbrace{\pi_p(\rho_p)}_{smooth.}$$

From now on, assume for simplicity ρ_p to be <u>absolutely irreducible</u>.

To make precise Breuil's idea, one first has to say which kind of reps we want to look at.

<u>Def</u> I.7 A unitary Banach space rep of $G(\mathbb{Q}_p)$ is an L-Banach space Π with a C^0 action of $G(\mathbb{Q}_p)$, and a norm $\|.\|$ inducing its topology which is $G(\mathbb{Q}_p)$-invariant ($\|gv\| = \|v\|$ $\forall g \in G(\mathbb{Q}_p)$).

<u>Def</u> I.8: Let Π unitary Banach rep, $\|.\|$ as before,

$$\Theta := \{v \in \Pi, \|v\| \leq 1\} \quad \text{its unit ball.}$$

$\Theta \otimes_{\mathcal{O}_L} k_L$ is a k_L-rep which is smooth (because $G(\mathbb{Q}_p)$-act is C^0)

Say Π is <u>admissible</u> if this k_L-rep is admissible in the usual sense ($\forall H < G(\mathbb{Q}_p)$ compact open, $\dim((\Theta \otimes_{\mathcal{O}_L} k_L)^H) < \infty$.)

Let $\mathrm{Ban}(G)$ be the category of unitary admissible Banach reps of $G(\mathbb{Q}_p)$.

Schneider-Teitelbaum: Π is admissible iff Π' is fini generated as a module over $L[\![G(\mathbb{Z}_p)]\!] = \underbrace{O_L[\![G(\mathbb{Z}_p)]\!]}_{\text{noetherian (Lazard)}} \otimes_{O_L} L$. Hence $\mathrm{Ban}(G)$ is an abelian category.

There is a simple way to associate to any rep. such as $\pi^{alg}(\rho_p)$ a unitary Banach rep of $G(\mathbb{Q}_p)$, called its <u>universal unitary completion</u>, and this is what I'll ~~to~~ now explain.

<u>Def 1.9.</u> V locally convex top L-vs, with a C^0 $G(\mathbb{Q}_p)$-action.
U unitary Banach rep of $G(\mathbb{Q}_p)$. Say that a given C^0 linear map G-eq $V \to U$ realizes U as a universal unitary completion of V

if $\mathrm{Hom}^{C^0}_{G(\mathbb{Q}_p)}(V, W) \xleftarrow{\simeq} \mathrm{Hom}(U, W)$ $\forall$ unitary Banach rep W.

(U is unique up to unique iso), call it $\hat{V}$.

(<u>Prop 1.10</u>) V admits a unitary universal completion iff the set of commensurab. classes of G-inv open lattice in V has a minimal element.
(<u>Pf</u>: if $\hat{V}$ exists, take the preimage of the open unit ball of $\hat{V}$, ...)
in the other direction, complete wrt to the gauge of the lattice)

This construction is simple & might look fairly nice, but in practice it turns out to be really bad:

- $\hat{V}$ does not always exist.
- $\hat{V}$ can be zero.
- $\hat{V}$ can be huge (not admissible for example).

<u>Ex</u>: Assume V to be top irr. If $\hat{V}$ exists and $\hat{V} \neq 0$, the map $V \to \hat{V}$ has dense image so it $\neq 0$, and hence must be injective. Pulling back a $G(\mathbb{Q}_p)$-inv norm on $\hat{V}$ to V, we see that V admits a $G(\mathbb{Q}_p)$-inv norm. It implies that the central character of V must be <u>unitary</u>.

Here is a criterion for $\hat{V}$ to exist :

Prop I.10. (...)

We can apply this to show that u.u.c always exist for fin. gen G-rep equipped with their finest loc. convex. top. :

Prop I.11 : V finitely generated G-rep with its finest loc. conv top.

The completion of V wrt to any finite type lattice of V ($= G(\mathcal{O}_p)$-inv lattice which is finitely generated as an $\mathcal{O}_L[G]$-module) is a u.u.c of V.

(Pf : Any lattice in V is open (finest loc. conv. top) & any G-inv lattice in V contains a finite type sublattice (since V is fin. gen).

Then apply prop I.10.)

Thus in our case, $\pi^{alg}(\rho_p)$ always exists. But it can very well be 0 !

(if $v_1, \dots, v_n$ is a gen. family , $\sum \mathcal{O}_L[G(\mathcal{O}_p)] v_i = \Theta$ can contain a L-line).

Also, it can be very huge. So this local pseudo-solution does not lead anywhere... except in the particular case of crystalline reps, as we will see in the next lectures.

A global solution for global representations :

Let $$\hat{H}(K^p) = \left(p\text{-adic completion of } \varinjlim_{K_p} H^1(Y(K^pK_p), \mathcal{O}_L)\right) \otimes_{\mathbb{Z}} L$$

$$\underset{\text{(same as)}}{=} \left(\varprojlim_n \varinjlim_{K_p} H^1(Y(K_pK^p), \mathcal{O}_L/\pi_L^n)\right) \otimes_{\mathcal{O}_L} L.$$

Endow it with the top which makes $\varinjlim_{K_p} H^1(Y(K_pK^p), \mathcal{O}_L)$ the unit ball.

It's a unitary Banach rep.

Th I.12 (Emerton, Ohta) $\hat{H}(K^p) \in \mathrm{Ban}(G(\mathbb{Q}_p))$.

Th I.13 : $\exists$ $\mathcal{G}_{\mathbb{Q}} \times G(\mathcal{O}_p)$-eq embedding $\bigoplus_W W^* \otimes H_W(K^p) \hookrightarrow \hat{H}(K^p)$

(the direct sum is above all irred k-alg reps of G)

(This thm is an incarnation of the fact that there exist many congruences between wt k & wt 2 modular forms if one allows the level at p to increase. Technically, one shows: $\hat{H}_W(K^p) \simeq \hat{H}(K^p) \otimes W$, & the proof is very simple: $\forall n \geq 1$,

obvious def (choose a lattice W_0 in W, etc.)

$H^1(Y(K_pK^p), W_0/p^n) \simeq H^1(Y(K_pK^p), \mathbb{Z}/p^n) \otimes W$

for K_p small enough (s.t. K_p acts triv on W_0/p^n))

Let's go back to our ρ attached to a modular form. Thm <u>I.1</u> gives an embedding of $G(\mathbb{Q}_p)$-repr:

$$\pi_p(\rho_p) \hookrightarrow \mathrm{Hom}_{\mathcal{G}_\mathbb{Q}}(\rho, H_k(K^p)) \quad \text{for } K^p \text{ small enough.}$$

Hence by <u>I.13</u>, get an embedding of $G(\mathbb{Q}_p)$-repr:

$$\pi^{alg}(\rho_p) \overset{(*)}{\hookrightarrow} \mathrm{Hom}_{\mathcal{G}_\mathbb{Q}}(\rho, \hat{H}(K^p)) \leftarrow \in \mathrm{Ban}(G)$$

(I.12 + closed by sub)

By pull back, get a $G(\mathbb{Q}_p)$-stable lattice in $\pi^{alg}(\rho_p)$.

Complete wrt this lattice ($\Leftrightarrow$, take the closure of the image of $(*)$)

Yields an interesting rep of $G(\mathbb{Q}_p)$, unitary admissible.

<u>BUT</u>: it's really hard to say anything directly about this rep! Moreover, & more importantly, what to do for repr of $\mathcal{G}_{\mathbb{Q}_p}$ which are not restrictions of global modular rep? (There are many of them, even ~~non~~ de Rham!).

We would like to have a purely local recipe, associating to ANY p-adic 2-dim'l rep of $\mathcal{G}_{\mathbb{Q}_p}$ a rep in $\mathrm{Ban}(G)$, which allows to recover the original Galois rep. The miracle of p-adic LL for $GL_2(\mathbb{Q}_p)$ is that such a thing exists & that's what I'll try to explain in the next lectures: Colmez constructed a functor:

$$\left\{ \begin{array}{c} \Pi \in \mathrm{Ban}(G) \\ \text{of finite length} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{finite dim'l} \\ \mathbb{Q}\text{-rep of } \mathcal{G}_{\mathbb{Q}_p} \end{array} \right\}$$

inducing a bijection:

$$\left\{ \begin{array}{c} \text{abs. irred} \\ \text{non ordinary} \end{array} \Pi \in \mathrm{Ban}(G) \right\} \underset{\Pi(V) \leftarrow V}{\overset{\Pi \mapsto V(\Pi)}{\longleftrightarrow}} \left\{ \begin{array}{c} \text{2-dim'l abs irred} \\ \text{cont's } L\text{-rep of } \mathcal{G}_{\mathbb{Q}_p} \end{array} \right\}$$

s.t. if $\rho : \mathcal{G}_{\mathbb{Q}} \to GL_2(L)$ is modular as before,

$$\mathrm{Hom}_{\mathcal{G}_{\mathbb{Q}_p}}(\rho, \hat{H}(K^p)) \simeq \Pi(\rho_p)^{\oplus m(K^p)}$$

with the same $m(K^p)$ as in I.1.

Note that the correspondence works even for non de Rham rep (very useful)

The goal of this course will be to give an idea of the way these constructions work, focusing on the case of de Rham Galois representations).