# LECT. II : $(\varphi,\Gamma)$-MODULES & p-ADIC HODGE THEORY

We want to study $\mathbb{C}^\circ$ $p$-adic rep of $\mathcal{G}_{\mathbb{Q}_p}$. Moreover, we don't want to focus only on de Rham ones: it's important to be able to work with families of representations.
A very powerful tool for studying these rep is Fontaine's theory of $(\varphi,\Gamma)$-modules, which replaces Galois rep by much more concrete objects.

Goal of <u>this</u> lecture: summary of the theory of $(\varphi,\Gamma)$-modules; its relation with $p$-adic Hodge theory.

<u>Notation</u>: $L/\mathbb{Q}_p$ finite ext (coefficient field, always assumed to be "big enough")

$$\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \underset{\chi}{\overset{\sigma}{\xrightarrow{\sim}}} \mathbb{Z}_p^\times \qquad H = \mathcal{G}_{\mathbb{Q}_p(\mu_{p^\infty})}$$
(cyclotomic)

<u>Def II.1</u>: If $R$ is a topological ring with $\mathcal{C}^\circ$ commuting actions of $\varphi,\Gamma$, a $(\varphi,\Gamma)$-module over $R$ is a finite free $R$-module with commuting $\mathcal{C}^\circ$ semi-linear actions of $\varphi$ and $\Gamma$.

<u>Examples of ring</u> $R$:

- $\mathcal{R}$ = Robba ring = germs of analytic functions near the unit circle $|T|=1$
$$= \left\{ \sum_{n\in\mathbb{Z}} a_n T^n,\ a_n \in L,\ \text{converges in some annulus } r \le |T| < 1 \right\}$$
$$\varphi(T) = (1+T)^p - 1 \qquad \sigma_a(T) = (1+T)^a - 1.$$
- $\mathcal{E}^\dagger$ = "overconvergent elements in $\mathcal{R}$" = bounded elements in $\mathcal{R}$ (it's a field)
- $\mathcal{E}$ = $p$-adic completion of $\mathcal{E}^\dagger$ (completion wrt the Gauss norm $\|f\| = \sup_n |a_n|$)

Its ring of integers (unit ball for the Gauss norm) is $\mathcal{O}_\mathcal{E} = \left\{ \sum a_n T^n,\ a_n \in \mathcal{O}_L\ \forall n,\ a_n \to 0\ \ n \to -\infty \right\}$

$$\mathcal{E} = \mathcal{O}_\mathcal{E}\left[\tfrac{1}{p}\right] \qquad \mathcal{O}_\mathcal{E} \text{ is a DVR with res. field } k_\mathcal{E} = k_L((T)).$$

natural topology on $\mathcal{O}_\mathcal{E}$ = weaker one s.t. $\mathcal{O}_\mathcal{E} \to k_\mathcal{E}$ is $\mathcal{C}^\circ$ when $k_\mathcal{E}$ has the $T$-adic topology. Topology on $\mathcal{E}$ = $\varinjlim_n$ (topology on $p^{-n}\mathcal{O}_\mathcal{E}$).

- $\mathcal{R}^+$ = ring of power series converging in the open unit disk

As for $\mathcal{R}$, endow it with the top defined by the norms $\|\cdot\|_\rho = \sup |a_n|\rho^n \quad 0<\rho<1$

**Def II.2**: Say $f \in R^+$ has order $\leq r$ if $(p^{-nr}\|f\|_{\rho^{1/p^n}})_n$ is bounded for some $0<\rho<1$. ($\Leftrightarrow$ any). E.g. $\log(1+T)$ has order $\leq 1$.

- $\mathcal{E}^+$ = element of $R^+$ with bounded coeff $= L \otimes \mathcal{O}_L[[T]]$.

**Th II.3** (Fontaine) One has equivalences of categories:

$$\{\mathcal{O}_L\text{-rep of } \mathcal{G}_{\mathbb{Q}_p}\} \leftrightarrow \varphi\Gamma^{\text{ét}}(\mathcal{O}_{\mathcal{E}})$$
$$\{L\text{-reps of } \mathcal{G}_{\mathbb{Q}_p}\} \leftrightarrow \varphi\Gamma^{\text{ét}}(\mathcal{E})$$

(A $(\varphi,\Gamma)$-module over $\mathcal{O}_{\mathcal{E}}$ is said to be étale if the linearization of $\varphi$ is an isomorphism (if $\varphi(D)$ generates $D$). A $(\varphi,\Gamma)$-module $D$ over $\mathcal{E}$ is étale if $\exists$ $(\varphi,\Gamma)$-stable $\mathcal{O}_{\mathcal{E}}$ lattice in $D$ which is an étale $(\varphi,\Gamma)$-module over $\mathcal{O}_{\mathcal{E}}$).

Sketch of proof: Fontaine-Wintenberger's theory of the field of norms shows that $H \simeq \mathcal{G}_E$, $E \simeq \mathbb{F}_p((T))$. Let's try first to understand $k_L$-reps of $\mathcal{G}_E$.

For such a rep $V$, let: $D(V) := (V \otimes_{k_L} k_{\mathcal{E}}^{sep})^{\mathcal{G}_E}$.

By Hilbert 90, $D(V) \otimes_{k_{\mathcal{E}}} k_{\mathcal{E}}^{sep} \simeq V \otimes_{k_L} k_{\mathcal{E}}^{sep}$. ($\varphi$-equivariant)

Thus, $D(V)$ is finite dim'l over $k_{\mathcal{E}}$ ($\dim_{k_{\mathcal{E}}} D(V) = \dim_{k_L} V$) and $\varphi: x\otimes v \mapsto x^p \otimes v$ gives after linearization an isomorphism of $D(V)$. Inverse $V(D) = (D \otimes k_{\mathcal{E}}^{sep})^{\varphi=1}$

This way, we get an equivalence $\{k_L\text{-rep of } \mathcal{G}_E\} \leftrightarrow \varphi\Gamma^{\text{ét}}(k_{\mathcal{E}})$.

Can try to lift this to study $\mathcal{O}_L$-reps of $\mathcal{G}_E$.

Let $A$ = strict henselization of $\mathcal{O}_{\mathcal{E}}$. It's a DVR of with res. field $k_{\mathcal{E}}^{sep}$.

~~Def II.4~~: If $V$ is an $\mathcal{O}_L$-rep of $\mathcal{G}_E$, set
$$D(V) = (A \otimes_{\mathcal{O}_L} V)^{\mathcal{G}_E}$$

The actions of $\varphi$ and $\mathcal{G}_E$ lift to $A$.

One shows that this gives an equivalence: $\{\mathcal{O}_L\text{-rep of } \mathcal{G}_E\} \leftrightarrow \varphi^{\text{ét}}(\mathcal{O}_{\mathcal{E}})$.

(with inverse $D \mapsto V(D) = (A \otimes_{\mathcal{O}_L} V)^{\varphi=1}$)

Actually, $\mathcal{G}_{\mathbb{Q}_p}$ acts on $A$: not obvious at all how it acts with this def.

action on $k_{\mathcal{E}}^{sep}$

Then one has that the two functors:
$$V \mapsto D(V) = (A \otimes_{\mathcal{O}_L} V)^H$$
$$D \mapsto V(D) = (A \otimes_{\mathcal{O}_L} V)^{\varphi=1}$$
are inverse equivalences.

Ex: Describe the étale rk 1 $(\varphi,\Gamma)$-module attached to a character $\mathcal{G}_{\mathbb{Q}_p} \to L^\times$.

The operator $\psi$: One has $\mathcal{E} = \bigoplus_{i=0}^{p-1} ((1+T)^i \varphi(\mathcal{E}))$.

So the same stays true for any $D \in \Phi\Gamma^{\text{ét}}(\mathcal{E})$: $D = \bigoplus_{i=0}^{p-1} ((1+T)^i \varphi(D))$.

$\Rightarrow \exists!$ left inverse $\psi$ of $\varphi$ st $\operatorname{Ker}\psi = \bigoplus_{i=1}^{p-1} (1+T)^i \varphi(D)$.

This operator will play an important role in the next lecture (see exercise below for one important prop of $\psi$: it "increases convergence").

More about the rings $\mathcal{E}^+$, $\mathcal{R}^+$:

let $\mathcal{D}_0(\mathbb{Z}_p, L)$ = measures on $\mathbb{Z}_p$ with values in $L = \mathcal{C}^0(\mathbb{Z}_p, L)^*$

$\mathcal{D}(\mathbb{Z}_p, L)$ = distributions ___ $= LA(\mathbb{Z}_p, L)^*$.

($\mathcal{D}_0(\mathbb{Z}_p, L) \subset \mathcal{D}(\mathbb{Z}_p, L)$ but not the same top)

To a measure $\mu$, we associate its Amice transform:

$$A_\mu(T) := \int_{\mathbb{Z}_p} (1+T)^x \mu(x) \in L[\![T]\!].$$

Prop II.5: The Amice transform induces an isomorphism:

$$\mathcal{D}_0(\mathbb{Z}_p, L) \simeq \mathcal{E}^+$$

(easy consequence of Mahler expansion of $\mathcal{C}^0$ f° on $\mathbb{Z}_p$)

Rk: (a) Topological iso if:

weak top on $\mathcal{E}^+$ and weak top (as a dual space) on $\mathcal{D}_0(\mathbb{Z}_p, L)$

p-adic top on $\mathcal{E}^+$ and strong top on $\mathcal{D}_0(\mathbb{Z}_p, L)$.

(b) Isom of algebras if uses convolution on LHS to define the product.

Prop II.6: The map $f \mapsto \phi_f : x \mapsto \operatorname{res}_0\left((1+T)^x f(T) \frac{dT}{1+T}\right)$ induces an isomorphism $\mathcal{E}/\mathcal{E}^+ \simeq \mathcal{C}^0(\mathbb{Z}_p, L)$.

Actually the two prop II.6, II.7 combine to give an exact sq:

$$0 \to \mathcal{D}_0(\mathbb{Z}_p, L) \xrightarrow{A} \mathcal{E} \xrightarrow{\phi} \mathcal{C}^0(\mathbb{Z}_p, L) \to 0$$

Rk: Everything stays true if replace $\mathcal{E}^+$ by $\mathcal{R}^+$ and $\mathcal{D}_0$ by $\mathcal{D}$, $\mathcal{C}^0$ by $LA$. Have an exact sequence: $0 \to \mathcal{D}(\mathbb{Z}_p, L) \to \mathcal{R} \to LA(\mathbb{Z}_p, L) \to 0$.

The actions of $\varphi, \Gamma, \psi$ on $\mathcal{E}^+$ / $\mathcal{R}^+$ become on $\mathcal{D}_0(\mathbb{Z}_p, L)$ / $\mathcal{D}(\mathbb{Z}_p, L)$:

$\forall f$ $\int_{\mathbb{Z}_p} f(x)\, \varphi(\mu)(x) = \int_{\mathbb{Z}_p} f(px)\, \mu(x)$ $\qquad \int_{\mathbb{Z}_p} f(x)\, \sigma_a(\mu)(x) = \int_{\mathbb{Z}_p} f(ax)\, \mu(x)$.

$$\int_{\mathbb{Z}_p} f(x)\, \psi(\mu)(x) = \int_{p\mathbb{Z}_p} f(p^{-1}x)\, d\mu(x).$$

<u>EX</u>: (a) Show that $A_{x\mu} = \partial A_\mu$, $\partial = (1+T)\frac{d}{dT}$.

(b) Show that $A_{\mathrm{Res}_{b+p^n\mathbb{Z}_p}(\mu)}(T) = p^{-n} \sum_{\eta \in \mu_{p^n}} \eta^{-b} A_\mu((1+T)\eta - 1)$.
$b \in \mathbb{Z}_p, n \geq 0$

In particular, $\mathrm{Res}_{\mathbb{Z}_p^*} = 1 - \varphi\circ\psi$.

(c) Show that the map: $\mu \mapsto \left(A_{\mathrm{Res}_{\mathbb{Z}_p}(\varphi^n(\mu))}\right)_{n\geq}$ induces an isomorphism between $\mathcal{D}_0(\mathbb{Q}_p, L)$ $(:= \mathcal{C}_c^0(\mathbb{Q}_p, L)^*)$, with the obvious def on $\varphi$ on this space, and $\varprojlim_\psi \mathcal{E}^+$.

$\leadsto$ This functional analytic interpretation of the "period rings" $\mathcal{E}^+$, $\mathcal{R}^+$ will play a fundamental role in the following.

One would like to study Galois rep thru $p$-adic analysis, but the series in $\mathcal{E}$ do not converge anywhere. The situation is much better with the rings $\mathcal{E}^\dagger$, $\mathcal{R}$. The following deep theorem shows that one can use these rings instead:

<u>Th</u> <u>II</u>.4: (Cherbonnier-Colmez, Berger, Kedlaya):

The categories $\varphi\Gamma^{\text{ét}}(\mathcal{E})$, $\varphi\Gamma^{\text{ét}}(\mathcal{E}^\dagger)$, $\varphi\Gamma^{\text{ét}}(\mathcal{R})$ are all equivalent.

<u>Rk</u>: (a) For this to make sense, one needs to define $\varphi\Gamma^{\text{ét}}(\mathcal{E}^\dagger)$, $\varphi\Gamma^{\text{ét}}(\mathcal{R})$: say that a $(\varphi,\Gamma)$-module $D^\dagger$ over $\mathcal{E}^\dagger$ is étale if $D = D^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{E}$ is. Say that a $(\varphi,\Gamma)$-module $D_{\mathrm{rig}}$ over $\mathcal{R}$ is étale if $\exists\, D^\dagger$ $(\varphi,\Gamma)$-module over $\mathcal{E}^\dagger$, étale, st. $D_{\mathrm{rig}} = D^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R}$.
(Kedlaya: other interpretation of $\varphi\Gamma^{\text{ét}}(\mathcal{R})$ as slope 0 objects of $\varphi\Gamma(\mathcal{R})$).

(b) The functors $\varphi\Gamma^{\text{ét}}(\mathcal{E}^\dagger) \to \varphi\Gamma^{\text{ét}}(\mathcal{E})$ giving the eq are the obvious ones:
$\varphi\Gamma^{\text{ét}}(\mathcal{E}^\dagger) \to \varphi\Gamma^{\text{ét}}(\mathcal{R})$ $\qquad D^\dagger \mapsto D^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{E}$, $D^\dagger \mapsto D^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R}$.

(c) It's really hard in practice to recover $D$ from $D_{\mathrm{rig}}$.

<u>EX</u>: Classify rank 1 $(\varphi,\Gamma)$-modules (étale or not) over $\mathcal{R}$.

Useful fact: let $\mathcal{E}^{]0,r_n]}$ = ring of rigid analytic functions on the annulus $|\zeta_{p^n}-1| \leq T < 1$.

By def, $\mathcal{R} = \varinjlim_n \mathcal{E}^{]0,r_n]}$.

$\mathcal{E}^{]0,r_n]}$ is stable under $\Gamma$, and $\varphi(\mathcal{E}^{]0,r_n]}) \subseteq \varphi(\mathcal{E}^{]0,r_{n+1}]}$.

$\forall D_{rig} \in \varphi\Gamma(\mathcal{R})$, can write: $D_{rig} = \varinjlim_n D^{]0,r_n]}$

with $D^{]0,r_n]}$ finite free $\mathcal{R}^{]0,r_n]}$-module stable under $\Gamma$, of $rk = rk_{\mathcal{R}} D_{rig}$,

and s.t. $\varphi(D^{]0,r_n]}) \subseteq D^{]0,r_{n+1}]}$, $\forall n \gg 0$.

So now, we are working with modules over rings of functions on annuli, and we'll see in the next § why it's so useful.

Ex: (difficult, see Colmez, La série principale unitaire, Prop 1.1.2):

Let $r>0$, $\alpha \in L$ st $v_p(\alpha)<0$. Show that if $(x_n)_n \in (\mathcal{E}^{]0,r]})^{\mathbb{N}}$ is a bounded sequence st $\varphi(x_{n+1}) = \alpha x_n$ $\forall n$, then $x_n \in \mathcal{R}^+$ $\forall n$.

$(\varphi,\Gamma)$-modules & p-adic HT. $D_{rig}(V)$ knows ev. about $V$. How to recover $D_{cris}$, $D_{dR}$ from $D_{rig}$?

$V$ any p-adic rep of $\mathcal{G}_{\mathbb{Q}_p}$.

Th II.7 (Berger) $D_{cris}(V) = (D_{rig}(V)[\frac{1}{t}])^{\Gamma}$. (hard thm!)

(as $\varphi$-modules over $L$)

Moreover if $V$ is crystalline, $\mathcal{R}[\frac{1}{t}] \otimes_L D_{cris}(V) \simeq \mathcal{R}[\frac{1}{t}] \otimes_{\mathcal{R}} D_{rig}(V)$.

What about $D_{dR}(V)$ and its filtration?

Let $f \in \mathcal{E}^{]0,r_n]}$. $f(\zeta_{p^n}-1) \in F_n := \mathbb{Q}_p(\mu_{p^n})$ makes sense. ("localization morphism at $\zeta_{p^n}-1$")

Can define $f(\zeta_{p^n} e^{t/p^n} - 1) \in F_n[[t]]$. Also makes sense; call it $\varphi^{-n}(f)$.

Any power series in $F_n[[t]]$ converges in $B^+_{dR}$ (because $t$ is a uniformizer of $B^+_{dR}$) (and $\overline{\mathbb{Q}}_p \subset B^+_{dR}$ by Hensel). Get a map:

"$\log[\varepsilon]$"

$$\begin{array}{ccc} \mathcal{E}^{]0,r_n]} & \xrightarrow[\mathcal{G}_{\mathbb{Q}_p}\text{-eq}]{C^0} & B^+_{dR} \\ & \searrow \quad F_n[[t]] \quad \nearrow & \end{array}$$

In general can define for any $D \in \varphi\Gamma^{ét}(\mathcal{R})$ a $C^0$ embedding

$D^{]0,r_n]} \hookrightarrow (B^+_{dR} \otimes V)^H$

Define: $D^+_{dif,n} = F_n[[t]] \otimes_{\varphi^{-n}, \mathcal{E}^{]0,r_n]}} D^{]0,r_n]} \xrightarrow[\text{still injective}]{\Gamma} (B^+_{dR} \otimes V)^H$

$\Gamma$ acts on $D^+_{dif,n}$.

Th II.8 (Fontaine, Cherbonnier-Colmez)

$V$ de Rham. $\forall i \in \mathbb{Z}$, $\mathrm{Fil}^i(D_{dR}(V)) = (t^i D^+_{dif,n})^\Gamma \subset D_{dR}(V)$.

(And $D_{dif,n} = D^+_{dif,n}[\frac{1}{t}] = F_n((t)) \otimes D_{dR}(V)$).

In the other direction, how to reconstruct $D_{rig}(V)$ ($\cap D^\dagger(V), D(V)$) from $D_{cris}, D_{dR}$?

Assume for simplicity that $V$ is crystalline. Th II.7 gives:

$$\mathcal{R}[\tfrac{1}{t}] \otimes_L D_{cris}(V) \simeq \mathcal{R}[\tfrac{1}{t}] \otimes_{\mathcal{R}} D_{rig}(V).$$

So, if we know $D_{cris}(V)$, we can find $\mathcal{R}[\frac{1}{t}] \otimes_{\mathcal{R}} D_{rig}(V)$.

Now, how to recover $\mathcal{R}$ from $\mathcal{R}[\frac{1}{t}]$?

Prop II.9. Let $f \in \mathcal{R}$. Then $f(\zeta_{p^n} - 1) = 0$ $\forall n \gg 0$ $\Leftrightarrow$ $t | f$ in $\mathcal{R}$.

As $f(\zeta_{p^n} - 1) = \varphi^{-n}(f) \bmod t$, the condition says that $\varphi^{-n}(f) \in t F_n[[t]]$, $\forall n \gg 0$.

Th II.10 (Berger) $V$ crystalline

$$D_{rig}(V) = \{ z \in \mathcal{R}[\tfrac{1}{t}] \otimes_L D_{cris}(V),\ \varphi^{-n}(z) \in D^+_{dif,n}\ \forall n \gg 0 \}$$

$$= \{ z \in \mathcal{R}[\tfrac{1}{t}] \otimes_L D_{cris}(V),\ \varphi^{-n}(z) \in \mathrm{Fil}^0(F_n((t)) \otimes D_{cris}(V)) \ \forall n \gg 0 \}$$

Rk: (a) For any filtered $\varphi$-module, the RHS of Th II.10 give a recipee to construct a $(\varphi,\Gamma)$-module over $\mathcal{R}$. Berger checks that the filtered $\varphi$-module is admissible (Lect I) iff the associated $(\varphi,\Gamma)$-module is étale. This gives a beautiful simple proof of Colmez-Fontaine's theorem (Th I.4).

(b) Everything extends to the potentially semi-stable (= de Rham) case, using $\mathcal{R}[\frac{1}{t}, \log T]$ and the Galois action.

Th II.11 (Berger)

$V$ crystalline. Choose a basis $(e_1, e_2)$ of $D_{cris}(V)$ s.t. $e_i$ is an eigenvector of $\varphi$. Then $z = z_1 \otimes e_1 + z_2 \otimes e_2 \in \mathcal{R}[\frac{1}{t}] \otimes_L D_{cris}(V)$ is in $D^\dagger(V)$ iff $\varphi^{-n}(z) \in \mathrm{Fil}^0(F_n((t)) \otimes D_{cris}(V))$ and $z_i \in \mathcal{R}_{x_i}$, $x_i \leq -v_p(\lambda_i)$. ($\lambda_i$ eigenvalues of $\varphi$).

Nice example: Describe the $(\varphi,\Gamma)$-module over $\mathcal{R}$ attached to a supersingular elliptic curve.

(See Berger Astérique 2008, IV.2.8 (3)).