LECT V. LOCALLY ANALYTIC / ALGEBRAIC VECTORS IN THE p-ADIC LLC

Let $\Pi \in \mathrm{Ban}(G)$. Can define two subspaces of Π:

- $\Pi^{an} = \{v \in \Pi,\ g \mapsto gv$ locally analytic $\}$
- $\Pi^{alg} = \{v \in \Pi,\ g \mapsto gv$ locally polynomial $\}$

$\Big\}$ G-stable

The goal of this lecture is to describe these two subspaces and their significance in the p-adic LLC, for $\Pi \in \mathrm{Ban}(G)$, abs irred & non ordinary.

Locally analytic vectors.

Th V.1 $\Pi^{an} \subset \Pi$ is dense if $\Pi \in \mathrm{Ban}(G)$
(Schneider-Teitelbaum)

The natural topology on Π^{an} is not the topology induced from the one of Π but rather the subspace topology wrt the G-eq embedding: (stronger)

$$\Pi^{an} \hookrightarrow LA(G,V) \qquad v \mapsto (g \mapsto g \cdot v)$$

One checks that Π^{an} is a closed subspace of $LA(G,V)$, hence a space of compact type (locally convex tvs, which can be written as an inductive limit of Banach with injective compact transition maps).

* Rk: Π^{an} need not be irreducible if Π is. But Π of f. length $\Rightarrow \Pi^{an}$ of f.l. (Colmez 2015)

The following result shows that one does not lose any information by restricting to Π^{an}:

Th V.2 (Colmez-Dospinescu) For any $\Pi \in \mathrm{Ban}(G)^{fl}$, admitting a central character (eg. $\Pi \in \mathrm{Ban}(G)$ abs irred), Π is the UUC of Π^{an}.

This looks like a purely rep theoretic statement, but the proof uses p-adic LLC! The key ingredient is to describe Π^{an} in terms of (φ, Γ)-modules, as was done for Π in Lect IV. More precisely, one has:

Th V.3 (Colmez, Dospinescu) Take (D, δ) G-compatible. Set $\Pi = \Pi_\delta(D)$, $\check{\Pi} = \Pi_\delta(\check{D})$.

There exists a G-eq sheaf on $\mathbb{P}^1(\mathbb{Q}_p)$, denoted $U \mapsto D_{rig} \boxtimes U$, st: $D_{rig} \boxtimes_\delta \mathbb{Z}_p = D_{rig}$

& $D_{rig} \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)$ sits in an exact sequence:

$$0 \to (\check{\Pi}^{an})^* \to D_{rig} \boxtimes_\delta \mathbb{P}^1 \to \Pi^{an} \to 0.$$

($D_{rig} \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)$ can be defined as before, once one has shown that w_δ on $(D^{[0,r]})^{\psi=0}$ extends uniquely to a continuous involution of $(D^{]0,r]})^{\psi=0}$ $\forall r>0$ small enough.

As Π^{an} is a locally an. rep of G on a space of compact, $(\Pi^{an})^*$ is a module over $\mathcal{D}(K)$.
Actually, $(\Pi^{an})^* \simeq \Pi^* \otimes_{\Lambda[1/p]} \mathcal{D}(K)$.
For $m \geq 2$, let $K_m = 1 + p^m M_2(\mathbb{Z}_p)$, a p-adic compact Lie group.

$\mathcal{D}(K_m)$ = distribution algebra of K_m
$= LA(K_m, L)^*$

$U(gl_2)$ = enveloping algebra of gl_2.
$\subseteq \mathcal{D}(K_m)$ (see elements of $U(gl_2)$ as diff operators on $LA(K_m, L)$ & evaluate at 1).

Prop: The action of G makes $D_{rig} \boxtimes \mathbb{P}^1$ a module over $\mathcal{D}(K_m)$. If $H \subset GL_2(\mathbb{Q}_p)$ compact open subgroup stabilizes $U \subset \mathbb{P}^1(\mathbb{Q}_p)$ compact open, $D_{rig} \boxtimes U$ is stable by $\mathcal{D}(H)$.

$\rightsquigarrow$ Induces an action of gl_2 on $D_{rig} \boxtimes \mathbb{Z}_p = D_{rig}$.

Let $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $u^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $u^- \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Thm V.4 (Dospinescu)

V 2-dim'l abs irred rep, with HT wts a, b, so that $P_{sen,V}(X) = (X-a)(X-b)$.
Then the action of gl_2 on D_{rig} is given by: $\forall z \in D_{rig}$,

$I_2 . z = (a+b-1) z$ $\qquad$ $h.z = 2\nabla(z) - (a+b-1) z$

$u^+ . z = t z$ $\qquad$ $u^- . z = -\dfrac{P_{sen,V}(\nabla)(z)}{t}$

where $\nabla = \lim\limits_{a \to 1} \dfrac{\sigma_a - 1}{a - 1}$ is the infinitesimal action of Γ and $t = \log(1+T) \in \mathcal{R}^+$.

Cor V.5: The Casimir element $C = u^+u^- + u^-u^+ + \frac{1}{2}h^2 \in Z(U(gl_2))$ acts on D_{rig} as multiplication by $\frac{(b-a)^2-1}{2}$.

Combining Cor V.5 & Thm IV.8, we get:

Cor V.6: Let $\Pi \in \mathrm{Ban}(G)$ abs. irred. Then Π^{an} admits an infinitesimal character.

Rk: (a) Even if Π is abs. irred, Π^{an} need not be abs irr, so this result cannot be deduced from Dospinescu-Schraen, for example.

(b) If $a, b \in L$, the space of rep V with generalized HT wts a, b has dim 3. The Banach rep associated to these rep all have the same inf character, by Cor V.5, but are not isomorphic. Hence there exist infinitely many $\Pi \in \mathrm{Ban}(G)$ abs irred with the same inf character, non isomorphic. This is in contrast with the theory of unitary reps of real semi-simple groups.

Pf of TH V.4: C commutes with the adjoint action of G hence C viewed as an operator on D_{rig} commutes with φ, Γ, and multiplication by $1+T$.

Hence $C(f(T)z) = f(T)C(z)$ $\forall z \in D_{rig}$, $f \in L[T]$. But $L[T]$ is dense in $\mathcal{R}$, hence $C \in \mathrm{End}_{\varphi,\Gamma,\mathcal{R}}(D_{rig}) = \mathrm{End}_{L[\mathcal{G}_{\mathbb{Q}_p}]}(V) = L$.

But the formulas for u^+, h (easy) give:

$$C = 2t\,u^- + 2P_{Sen,V}(\nabla) + \frac{(a-b)^2-1}{2}$$

hence $C - \frac{(a-b)^2-1}{2}$ sends D_{rig} to $t D_{rig}$ because $P_{Sen,V}(\nabla)$ does: indeed it suffices to show that $\varphi^{-n}(P_{Sen,V}(\nabla)(z)) \in t D^+_{dif,n}$ $\forall z \in D_{rig}$ $\forall n \gg 0$ i.e. that $P_{Sen,V}(\nabla)$ sends $D^+_{dif,n}$ into $t D^+_{dif,n}$. But by definition $P_{Sen,V}$ is the characteristic polynomial of ∇ on $D_{Sen,n} = D^+_{dif,n}/tD^+_{dif,n}$ hence it suffices to apply Cayley-Hamilton! $\square$

The output is that even if the action of G (w) is complicated, we can describe the action of gl_2!

Th V.7 (Dospinescu) Let $J(\pi) = \pi / \langle (u-1)v, v \in \pi, u \in \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \rangle$ naive Jacquet module

Let V 2-dim'l abs irr. Then $J(\Pi(V)) \neq 0 \iff V$ trianguline (Lect IV)

The $\Rightarrow$ direction is an applicat° of V.4.

It's a p-adic analogue of the fact that irreducible WD rep $\Leftrightarrow$ supercuspidal
(recall de Rham + trianguline $\Rightarrow$ $WD(D_{pst})$ reducible as $W_{\mathbb{Q}_p}$-rep)

Locally algebraic vectors

The main thm is

Th V.7: (Colmez, Dospinescu)

V abs irred 2 dim'l. $\Pi(V)^{alg} \neq 0 \Leftrightarrow V$ de Rham with HT wts distinct.

In this case, if $a < b$ are the HT wts,

$$\Pi(V)^{alg} \simeq LL(D_{pst}(V)) \otimes (\mathrm{Sym}^{b-a-1} \otimes \det{}^a)$$

This shows that the p-adic LLC "encodes" the classical LLC, and it confirms Breuil's philosophy that for de Rham rep V with HT wts $\neq$, $\Pi(V)$ should be a completion of the above locally alg. rep.

Sketch of the pf of the first part of Th V.7:

Let Y be a B-module with central char ω.

Def V.8 A Kirillov model for Y is the data of:

- $\overline{Y}$ a $L_\infty[[t]]$-module with an action of $\mathcal{D}(\Gamma)$
- a B-eq injection $Y \hookrightarrow \underbrace{LA_{rc}(\mathbb{Q}_p^*, \overline{Y})^\Gamma}_{\{\phi,\ \phi: \mathbb{Q}_p^* \to \overline{Y} \text{ loc an, with compact support in } \mathbb{Q}_p,\ \forall x\ \forall a\ \sigma_a(\phi(x)) = \phi(ax)\}}$

with B action:

$$\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \phi\right)(x) = \omega(d)\left[\varepsilon^{bx/d}\right]\phi\left(\frac{ax}{d}\right).$$

$(1+T = [\varepsilon])$.

$Y_c :=$ inverse image of $LA_c(\mathbb{Q}_p^*, \overline{Y})$.

Ex: (a) If Y is a smooth rep, take $\overline{Y} = L_\infty$ (viewed as $L_\infty[[t]]/t$). with its action of Γ. This gives the usual Kirillov model of smooth B-repr (note that the action of Γ on L_∞ is smooth, hence $LA_{rc}(--) = LC_{rc}(--)$)
& $Y/Y_c =$ Jacquet module.

(b) If $Y = Y' \otimes \mathrm{Sym}^{k-1}$, Y' smooth, we take $L_\infty[[t]]/t^k$ (space of L_∞ poly. functions of deg $\leq k-1$).

Let $\Pi^{u^+\text{-fin}} = \{ v \in \Pi^{an}, \ \exists k > 0 \quad (u^+)^k \cdot v = 0 \}$

It's a sub B-module of Π^{an}.

Let $\tilde{v}$ be a lifting of v in $D_{rig} \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)$.

$v \in \Pi^{u^+\text{fin}} \Rightarrow \exists N, k \in \mathbb{N} \quad \left(\begin{pmatrix} 1 & p^N \\ & 1 \end{pmatrix} - 1 \right)^k \cdot \tilde{v} \in (\check{\Pi}^{an})^*$.

The image of $(\check{\Pi}^{an})^*$ by $\mathrm{Res}_{\mathbb{Z}_p}$ live in $D^{]0, r(D)]}$ ($r(D)$ = radius of convergence of D) : indeed, $(\check{\Pi}^{an})^*$ is a Fréchet, and $D_{rig} \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)$ $= \varinjlim_r \underbrace{D^{]0,r]} \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)}_{\text{Fréchet}}$, so its image lands in $D^{]0,r]} \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)$, some r. Moreover ψ is surjective on this image, as $\begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix}$ is on $(\check{\Pi}^{an})^*$.

If $z \in (\check{\Pi}^{an})^*$, $\begin{pmatrix} p^j a & 0 \\ 0 & 1 \end{pmatrix} z \in (\check{\Pi}^{an})^* \quad \forall a \in \mathbb{Z}_p^\times, j \in \mathbb{Z}$.

$\Rightarrow \mathrm{Res}_{\mathbb{Z}_p} \left(\begin{pmatrix} p^j a & 0 \\ 0 & 1 \end{pmatrix} \cdot z \right) \in D^{]0, r(D)]} \quad \forall a, j$.

Set for $n \geq m(D)$,

$$\iota_n \left(\mathrm{Res}_{\mathbb{Z}_p} \begin{pmatrix} p^j a & 0 \\ 0 & 1 \end{pmatrix} \tilde{v} \right) = \frac{1}{\varphi^{N+j-n}(\sigma_a(T))^k} \iota_n \left(\mathrm{Res}_{\mathbb{Z}_p} \left(\begin{pmatrix} p^j a & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & p^N \\ 0 & 1 \end{pmatrix} - 1 \right)^k \tilde{v} \right) \right)$$
$$\in t^{-k} D^+_{dif, n}$$

(because $\varphi^q(T) \mid t$ in B^+_{dR}, $\forall q$)

One checks that the LHS does not depend on the choice of $\tilde{v}$, ~~nor on $n \geq m(D)$~~ after projection in $D^-_{dif,n}$. For $x \in \mathbb{Q}_p^\times$, define $\phi_v(x)$ to be the image of $\iota_n \left(\mathrm{Res}_{\mathbb{Z}_p} \begin{pmatrix} p^n x & \\ 0 & 1 \end{pmatrix} \tilde{v} \right)$ in D^-_{dif} : it does not depend on $\tilde{v}$, nor on $n \geq m(D)$.

<u>Prop</u> V.9 (Colmez) (i) $v \mapsto \phi_v \in LA_{rc}(\mathbb{Q}_p^\times, D^-_{dif,m})^\Gamma$ is a Kirillov model for $\Pi^{u^+\text{-fin}}$.

(ii) $U(\mathfrak{gl})$ stabilizes $\Pi^{u^+\text{fin}}$ and if $\lambda \in U(\mathfrak{g})$,

$$\phi_{\lambda \cdot v}(x) = \begin{pmatrix} x & \\ & 1 \end{pmatrix} \lambda \begin{pmatrix} x^{-1} & \\ & 1 \end{pmatrix} \cdot \phi_v(x)$$

The proof of (ii) uses <u>Th V.4</u> and is a direct computation.

Now, for Π^{alg} to be $\neq 0$, the central char. has to be loc. alg, i.e. $a+b$ must be an integer. If so, $v \in \Pi^{an}$ is in Π^{alg} iff v is $U(\mathfrak{sl}_2)$-finite. Such a vector v is in particular in $\Pi^{u^\pm\text{fin}}$.

If $\lambda \in \mathcal{U}(\mathfrak{sl}_2)$, $\phi_{\lambda v}(x) = \begin{pmatrix} x & \\ & 1 \end{pmatrix} \lambda \begin{pmatrix} x^{-1} & \\ & 1 \end{pmatrix} \phi_v(x)$ (Prop V-9)

So, if v is killed by a finite codim ideal I of $\mathcal{U}(\mathfrak{sl}_2)$, so is $\phi_v(x)$ (killed by $\begin{pmatrix} x & \\ & 1 \end{pmatrix} I \begin{pmatrix} x^{-1} & \\ & 1 \end{pmatrix}$) : in other words, ϕ_v takes values in $(\mathbb{D}_{dif}^-)^{\mathcal{U}(\mathfrak{sl}_2)\text{-fin}}$.

What does $(\mathbb{D}_{dif}^-)^{\mathcal{U}(\mathfrak{sl}_2)\text{-fin}}$ look like? Suppose that ∇ acts semisimply on D_{dif}^+. Then $\exists\, e_1, e_2$ basis of D_{dif}^+ over $L_\infty[[t]]$, st. $\nabla e_i = \lambda_i e_i$. ($\lambda_i \in \{a, b\}$)

Then $a^+(t^j e_i) = (\lambda_i + j) e_i$, $u^+(t^j e_i) = t^{j+1} e_i$

$u^-(t^j e) = -(\lambda_i + j - a)(\lambda_i + j - b)\, t^{j-1} e_i$

If $(D_{dif}^-)^{\mathcal{U}(\mathfrak{sl}_2)\text{-fin}}$ is $\neq 0$, its preimage in D_{dif} is a $L_\infty[[t]]$-lattice of D_{dif} stable by $\mathcal{U}(\mathfrak{sl}_2)$. But the above formulas show that there is such a non trivial ($\neq D_{dif}^+$) lattice iff $\exists\, j \neq 0$ st $(\lambda_i + j - a)(\lambda_i + j - b) = 0$. This happens only when $a - b \in \mathbb{N}-\{0\}$ (i.e. a, b distinct integers, as $a+b$ was assumed to be integer). In this case, one can check that V is de Rham with distinct HT wts. Hence $(D_{dif}^-)^{\mathcal{U}(\mathfrak{sl}_2)\text{-fin}}$ is $\neq 0$ iff V is dR with distinct HT wts.

So we see that if $\Pi^{alg} \neq 0$, V must be de Rham with distinct HTW. The converse is not difficult ($(D_{dif}^-)^{\mathcal{U}(\mathfrak{sl}_2)\text{-fin}}$ is finit dim'l, hence the annulator ideal I of this rep is stable by conjugation, ...). End of the sketch of pf of V.7. $\square$

"Application" : the Fontaine - Mazur conjecture (in many cases)

Let $\rho : \mathcal{G}_{\mathbb{Q}} \to GL_2(L)$ continuous abs ired, odd, unr. a.e., de Rham at p with $\neq$ HT wts. + some technical hypotheses on ρ : irreducibility of $\bar{\rho}$, ρ_p. F-M says that ρ is geometric / twist of a modular rep. (important)

The most natural approach to the conjecture would be by the Taylor-Wiles (-Kisin) method. But to control the deformation rings, you have to put some stronger conditions at p (crystalline, Font-Laffaille, ...). (singularities) However Kisin was able to control the singularities of def rings enough to prove F-M using this strategy.

Emerton's strategy is totally different, and completely avoids the study of complicated def rings.

Its proof has 3 steps:

Step 1: Show that ρ is promodular, meaning that it's associated to some maximal ideal (not necessarily classical) of some big Hecke algebra $\mathbb{T}$. Very roughly, this means that ρ comes from a "p-adic modular form". This is very much easier than showing modularity: it boils down to show that modular points are Zariski dense in the def space of a.e. unr def of $\bar{\rho}$ (no conditions at p) (indeed then $\mathrm{Spec}(\mathbb{T}) \to \mathrm{Spec}(R)$ is closed defined by ideal I, and dense, so I is nilpotent; as R is reduced, $I = 0$, ie $\mathbb{T} \simeq R$). This you can do by finding a modular point and looking at Coleman families passing thru this point.

Step 2: With notations of Lect I, show that for ρ as before + ~~abs irred~~ promodular.

<u>Thm V.10</u> $\Pi\mathbb{B}(\rho_p) \hookrightarrow \mathrm{Hom}_{\mathcal{G}_\mathbb{Q}}(\rho, \hat{H}^1(K^p))$ for K^p small enough.

This was mentioned in Lect I for modular ρ. But really the important hypothesis is that ρ is promodular. This statement shows that p-adic LL realizes in completed cohomology of modular curves! Emerton's proof of this thm is highly indirect (which is not surprising, given Colmez's constructions...) but very beautiful: you first show it when ρ comes from a modular form f with level prime to p (in this case, it's "easy": ρ_p is crystalline and we know that classical LLC realizes in the usual coh of modular curves (Th I.1); moreover the locally alg rep has at most one equivalence class of invariant norms as we saw in Lect III, etc.).

Step 3: Using Th V.7, and the assumption that ρ_p is dR with $\neq$ HT wts, get

$$\Pi(\rho_p)^{\mathrm{alg}} \neq 0 \underset{(\text{Step 2})}{\Rightarrow} \mathrm{Hom}_{\mathcal{G}_\mathbb{Q}}(\rho, \hat{H}^1(K^p))^{\mathrm{alg}} = \mathrm{Hom}_{\mathcal{G}_\mathbb{Q}}(\rho, \hat{H}(K^p)^{\mathrm{alg}}) \neq 0$$

Now, remember Thm I.13 of lect I:

$$\bigoplus_W W^* \otimes_L H_W(K^p) \hookrightarrow \hat{H}(K^p)$$

Actually, Emerton shows a better result:

$$\bigoplus_W W^* \otimes_L H_W(K^p) \simeq \hat{H}(K^p)^{alg}.$$

So we conclude with I.1 that ρ is modular (or directly that ρ is geometric).