LECT VI : p-ADIC LLC AND GEOMETRY: THE SEMI-STABLE CASE

Recall Thm $\underline{\mathrm{I}}.1$ of Lect I: for every $k \geq 2$, $H_k \; (:= \varinjlim_{K_f} H^1(Y(K_f), \mathrm{Sym}^{k-2}))$ decomposes as the direct sum of $\underset{G_{\mathbb{Q}}}{\overset{\circlearrowleft}{\rho_f}} \otimes \underset{GL_2(\mathbb{A})}{\overset{\circlearrowleft}{\pi(f)}}$, f primitive cusp form of wt k.

(ρ_f is the Galois rep attached to f, $\pi(f)$ the aut. rep attached to f)

(Actually Thm I.1 as stated was weaker, we didn't say anything about places $\neq p$).

Emerton's Thm $\underline{\mathrm{V}}.10$ (suitably formulated) is an astonishing generalization of this to p-adically completed coh.

However, these two results only dealt with global Galois representations, and we're only interested here by what happens at p. Are there geometric objects which would realize (classical or p-adic) LLC for all reps?

The answer to this question is well known in the classical case, and this is what I want to explain first. (and goes under the name of non abelian LT theory)

The $\underline{\text{Drinfeld tower}}$:

It's a tower of rigid analytic spaces over $\breve{\mathbb{Q}}_p$ (= completion of $\mathbb{Q}_p^{unr}$)

$$\begin{array}{l} \breve{M}_n \\ \downarrow \\ \vdots \qquad \text{étale Galois} \\ \downarrow \\ \breve{M}_0 \\ \downarrow \\ \breve{\Omega} = \Omega \hat{\otimes}_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p \end{array}$$

Ω = usual Drinfeld's half-plane whose $\mathbb{C}_p$-points are $\Omega(\mathbb{C}_p) = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$.

M_0 = rigid generic fiber of the RZ formal scheme of deformations by quasi-isogenies of a special formal O_D-module* of dim 2, ht 4 over $\overline{\mathbb{F}}_p$.

Drinfeld: $\breve{M}_0 \simeq \breve{\Omega} \times \mathbb{Z}$.

(Here $O_D \subset D$ unique max order of the non split q. alg $/\mathbb{Q}_p$.
$O_D = \mathbb{Z}_{p^2}[\Pi]$, $\Pi^2 = p$, $\underset{x \in \mathbb{Z}_{p^2}}{\Pi x = \sigma(x)\Pi}$)

$\mathcal{M}_n$ is obtained as usual by adding some level structures.

The geometry of $\mathcal{M}_n$ is much more complicated!

Have two actions on this tower:

* horizontal action of G ($= GL_2(\mathbb{Q}_p)$) (G is the group of self $\mathcal{O}_D$-quasi-isog of $\mathbb{X}$). The action of G on $\mathcal{M}_0$ is the usual action by homographies on the factor $\breve{\Omega}$ and shift by $v_p(\det(g))$ on $\mathbb{Z}$.

* vertical action of $D^\times$. Its action on $\mathcal{M}_n$ factors thru $D^\times/1+p^n\mathcal{O}_D$ (and the Galois group of $\mathcal{M}_n \to \breve{\Omega}$ is $D^\times/1+p^n\mathcal{O}_D$).

Before stating the next thm, recall that a Weil-Deligne rep is the same thing as an ℓ-adic (continuous) rep of the Weil group (because of Grothendieck's ℓ-adic monodromy thm). $\ell \neq p$

<u>Th VI.1</u>

Let $\ell \neq p$ prime, ρ smooth irred rep of $D^\times$ with triv. cc. Then

$$\mathrm{Hom}_{D^\times}\left(\rho, \varinjlim_n H^1_c\left(\mathcal{M}_n \hat{\otimes} \mathbb{C}_p/(p_p)^{\mathbb{Z}}, \overline{\mathbb{Q}}_\ell\right)\right) = \underbrace{\cancel{JL(\rho^\vee)}}_{\pi(\rho)^\vee} \otimes LL^{-1}(\pi(\rho))$$

with $\pi(\rho) = JL(\rho)$ (just to avoid connected components)

<u>Rk</u>: (a) This thm says that the ℓ-adic coh. of the Drinfeld tower realizes both the LLC & the JLC, for certain repr (see (b)).

(b) Recall that the (local) Jacquet-Langlands corresp. gives a "natural" bijection between $\left\{\begin{array}{l}\text{smooth irred}\\ \text{rep of } D^\times\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{smooth irred}\\ \underline{\text{square integrable}} \text{ repr of } G\end{array}\right\}$

square <u>integrable</u> = twist of the Steinberg or supercuspidal.

No such geometric realization of the local correspondence for principal series (but somehow square integ. repr are the 'building blocks').

(c) Thm <u>VI</u>.1 as stated is slightly incorrect: when $\rho = 1$, ie. when $\pi(\rho)$ is the Steinberg rep, the ρ-isotypic part of the cohomology does not recover $LL^{-1}(\pi(\rho))$: the monodromy is zero.

Goal of this lecture & the next one: describe a p-adic analogue of this thm. Today: the semi-stable (non crystalline) case.

If V is ss (non crys), $LL(WD(D_{st}(V)))$ is a twist of the Steinberg rep $St = LC(\mathbb{P}^1(\mathbb{Q}_p), L)/L$.

Let me first recall the description of V and $\Pi(V)$:

Up to twist, can assume HT wts $= 0, k$, $\det V = \chi^k$ (χ ?), and:

$$D_{st}(V) = Le_1 \oplus Le_2 \qquad Ne_1 = e_2,\ Ne_2 = 0 \qquad \varphi(e_2) = p^{-\frac{(k-1)}{2}} e_2 \quad \varphi(e_1) = p^{-\frac{(k+1)}{2}} e_1$$

$$\mathrm{Fil}^i(D_{st}(V)) = \begin{cases} D_{st}(V) & i \leq -k \\ L(e_1 - \mathcal{L} e_2) & -k < i \leq 0, \text{ some } \mathcal{L} \in L \\ 0 & i > 0 \end{cases}$$

(In term of $\mathscr{S}_{irr}$, $D(V) = D(\delta_1, \delta_2, \mathcal{L})$, with $\delta_1 = x^k |x|^{\frac{k+1}{2}}$, $\delta_2 = |x|^{\frac{k-1}{2}}$, $\delta_s = x^{k-1}$)

Then (Thm IV.11) $\Pi(V)$ is the quotient of:

$$D(k) := \left\{ f : \mathbb{Q}_p \to L,\ f(p\cdot)|_{\mathbb{Z}_p} \in \mathcal{C}^{\frac{k-1}{2}}(\mathbb{Z}_p, L),\ x \mapsto x^{k-1} f\left(\tfrac{1}{x}\right)|_{\mathbb{Z}_p - \{0\}} \text{ extends to an element of } \mathcal{C}^{\frac{k-1}{2}}(\mathbb{Z}_p, L) \right\}$$

(called $\hat{B}(V)$ last time)

($\mathcal{C}^{\frac{k-1}{2}}$ functions on $\mathbb{Q}_p$ with a pole of order $\leq k-1$ at ∞)

with G-action given by:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f\right)(x) = |ad - bc|^{\frac{k-1}{2}} (a - cx)^{k-1} f\left(\frac{dx - b}{a - cx}\right)$$

This space contains a G-stable subspace: the space generated by polynomials of degree $\leq k-1$ (rep $\simeq \mathrm{Sym}^{k-1}(L^2)$) & $h(x) = \sum_I \lambda_i (x - x_i)^{n_i} \log_{\mathcal{L}}(x - x_i)$

I finite, $\lambda_i \in L$, $x_i \in \mathbb{Q}_p$, $n_i \in \{\lfloor \frac{k-1}{2} \rfloor + 1, \dots, k-1\}$, $\deg\left(\sum_I \lambda_i (x - x_i)^{n_i}\right) < \frac{k-1}{2}$.

(Ex: do the computation of the action to understand why the case $\delta_s = x^{\text{integer} \geq 0}$ is special!)

Then: $\Pi(V) = D(k) / (\text{closure of this subspace})$.

Rk: $D(k)$, and even $D(k)/\mathrm{Sym}^{k-1}$ is not admissible when $k > 1$.

Colmez also described the locally analytic vectors of $\Pi(V)$: (in fact for any trianguline V)

For $\delta_1, \delta_2 : \mathbb{Q}_p^* \to L^*$, let:

$$B^{an}(\delta_1, \delta_2) = \mathrm{Ind}_B^{\mathrm{loc\,an}, G}(\delta_2 \otimes \delta_1 \chi^{-1}).$$

Let $(\delta_1, \delta_2, \mathcal{L}) \in S_{in}$ the parameter corresponding to V so not crystalline.
$B^{an}(\delta_1, \delta_2)$ contains a G-stable subspace isomorphic to $\underline{\mathrm{Sym}}^{k-1}(L^2)$. One has:

$$\mathrm{Ext}^1_{an}\left(\mathrm{Sym}^{k-1}(L^2), B^{an}(\delta_1,\delta_2)/\underline{\mathrm{Sym}}^{k-1}(L^2)\right) \simeq \mathrm{Hom}(\mathbb{Q}_p^\times, L).$$

Let $\Pi(k, \mathcal{L})$ be the loc an rep extension of Sym^{k-1} by $B^{an}(\delta_1,\delta_2)/\underline{\mathrm{Sym}}^{k-1}$ corresponding to $\mathcal{L}$ by this isomorphism. Then $\Pi(V)^{an}$ is an ext:

$$0 \to \Pi(k, \mathcal{L}) \to \Pi(V)^{an} \to B^{an}(\delta_2, \delta_1) \to 0$$

<u>Rk</u>. The UUC of $B^{an}(\eta_1, \eta_2)$ is zero when $v_p(\eta_2(p)) > 0$. (Colmez-Dospinescu, Prop 0.3). So in our case, the UUC of $\widehat{B^{an}(\delta_2, \delta_1)}$ is zero. We don't see it in the UUC, and in fact one has: $\Pi(V) \simeq \widehat{\Pi(k, \mathcal{L})}$.

Hence $\Pi(k, \mathcal{L})$ already knows everything about V. This is this loc an rep that we will describe geometrically.

For the rest of this lecture, set:

$$\Pi(k) := B^{an}(\delta_1, \delta_2)/\underline{\mathrm{Sym}}^{k-1}(L^2). \qquad \left(\underline{\mathrm{Sym}}^{k-1}(L^2) := \mathrm{Sym}^{k-1}(L^2) \otimes |\cdot|^{\frac{k-1}{2}}\right)$$

$B^{an}(\delta_1, \delta_2)$ can be seen as the space

$$D^{an}(k) := \left\{ f : \mathbb{Q}_p \to L,\ f_{|\mathbb{Z}_p} \in LA(\mathbb{Z}_p, L),\ x \mapsto x^{k-1} f\left(\tfrac{1}{x}\right)_{|\mathbb{Z}_p - \{0\}} \text{ extends to } LA(\mathbb{Z}_p, L) \right\}$$

with the same G-action as before; the subspace of polynomials of deg $\leq k-1$ is G-stable, $\simeq \underline{\mathrm{Sym}}^{k-1}(L^2)$.

Morita duality:

The group G acts on left by homographies on Ω, hence on the left on $\mathcal{O}(\Omega)$ by: $g \cdot f = f(g^{-1})$. For $k \in \mathbb{Z}$, let $\mathcal{O}(k)(\Omega)$ be the G-rep which is $\mathcal{O}(\Omega)$ as a t.v.s., with G-action twisted: (Assume k even for simplicity)

$$(g *_k f)(z) = |ad-bc|^{-\frac{k-2}{2}} (ad-bc)(a-cz)^{-k} f\left(\frac{dz-b}{a-cz}\right).$$

The space Ω is Stein: it ~~can be with~~ has an admissible increasing cover by affinoids $\mathcal{X}_n$ s.t. $\forall n$ $\mathcal{O}(\mathcal{X}_{n+1}) \xrightarrow{\text{restr.}} \mathcal{O}(\mathcal{X}_n)$ is compact with dense image. Hence $\mathcal{O}(k)(\Omega)$ is a C^0 rep of G on a Fréchet space.

<u>Rk</u>: If $k = 2$, $\mathcal{O}(2)(\Omega) \simeq \Omega^1(\Omega)$

Prop VI.2: (Morita, Schneider-Teitelbaum)

One has an isomorphism of G-repr:

$$\Pi(k) \simeq \mathcal{O}(k+1)(\Omega)^*.$$

"Proof": It's easy to check that the map I:

$$\ell \in \mathcal{O}(k+1)(\Omega)^* \mapsto I(\ell) : x \mapsto \ell\left(\frac{1}{z-x}\right)$$

has values in the space of loc an f^n on $\mathbb{Q}_p$ with a pole of order $\leq k-1$ at ∞, is continuous, G-eq, and injective (because the vs generated by the functions $z \mapsto \frac{1}{z-x}$, $x \in \mathbb{Q}_p$, is dense in $\mathcal{O}(\Omega)$). (Actually, I has values in the space of loc an f^n on $\mathbb{Q}_p$ which vanish at ∞, which is $\simeq \Pi(k)$ as a tvs).

To see that I is an iso, we only have to prove surjectivity, because of the open mapping thm; this is an explicit computation using residues along annuli. □

Rk. The transpose of the map I is J:

$$\Pi(k)^* \to \mathcal{O}(k+1)(\Omega), \quad \mu \mapsto \left(z \mapsto \int_{\mathbb{P}(\mathbb{Q}_p)} \frac{1}{z-x}\, d\mu(x)\right)$$

There is another way to prove Prop VI.2 (essentially the same proof!).

Chiarellotto: X Stein, $\dim X = 1$, $\mathcal{F}$ vb on X. Then:

$$H^1_c(X, \mathcal{F}) \simeq H^0(X, \mathcal{F}^\vee \otimes \Omega^1_X)^*.$$

Taking into account the G-action, this gives:

$$H^1_c(\Omega, \mathcal{O}(-k)) \underset{G\text{-eq}}{\simeq} \mathcal{O}(k+1)(\Omega)^*.$$

Moreover as Ω is Stein, the long exact sq

$$0 \to H^0_c(\Omega, \mathcal{O}(-k)) \to \mathcal{O}(-k)(\Omega) \to \varinjlim_Z \mathcal{O}(-k)(\Omega \setminus Z) \to H^1_c(\Omega, \mathcal{O}(-k)) \to H^1(\Omega, \mathcal{O}(-k))$$

reduces to a short exact sq: (vanishing of coh. coh + Serre duality)

$$0 \to \mathcal{O}(-k)(\Omega) \to \varinjlim_Z \mathcal{O}(-k)(\Omega \setminus Z) \to H^1_c(\Omega, \mathcal{O}(-k)) \to 0$$

whence

$$\mathcal{O}(k+1)(\Omega)^* \simeq \varinjlim_Z \mathcal{O}(-k)(\Omega \setminus Z) \Big/ \mathcal{O}(-k)(\Omega).$$

(Z finite union of admissible affinoids)

Can take $Z = Z_n$ s.t. $\Omega \setminus Z_n = \{ z \in \mathbb{P}^1(\mathbb{C}_p),\ d(z, \mathbb{P}^1(\mathbb{Q}_p)) < \frac{1}{n} \} \setminus \mathbb{P}^1(\mathbb{Q}_p)$ $(n>0)$

Let $f \in \mathcal{O}^{(k-1)}(\Omega \setminus Z_n)$. Can write $f = \lim_j f_j$, f_j linear combination of $(z-x)^i$, $x \in \mathbb{Q}_p$, $i \in \mathbb{Z}$.

For each j, write $f_j = \underbrace{f_j^+}_{\text{indices } i \geq 0} + \underbrace{f_j^-}_{\text{indices } i<0}$

$f_j^+ \to f^+$, $f_j^- \to f^-$

f^+ extends to a function on $\{ z \in \mathbb{P}^1(\mathbb{C}_p),\ d(z, \mathbb{P}^1(\mathbb{Q}_p)) < \frac{1}{n} \}$ whereas $f^- \in \mathcal{O}(\Omega)$. This shows that the quotient $\varinjlim_Z \mathcal{O}(-k)(\Omega \setminus Z) / \mathcal{O}(-k)(\Omega)$ is isomorphic as a t.v.s to $LA(\mathbb{P}^1(\mathbb{Q}_p), L) / \underbrace{\mathcal{O}(\mathbb{P}^1)}_{= L} = St^{an}$. $\square$

By def, on Z_n any $f \in \mathcal{O}(k)(\Omega)$ can be written

$$f(z) = \sum_{j=0}^{\infty} b_j z^j + \sum_{i=1}^{s} \sum_{j=1}^{\infty} \frac{b_{i,j}}{(z-x_i)^j}.$$

s integer, $|b_i| n^{-i} \xrightarrow[i \to \infty]{} 0$, $|b_{i,j}| n^i \xrightarrow[i \to \infty]{} 0$, $x_i \in \mathbb{Q}_p$, $b_j, b_{i,j} \in L$. $\mathbb{P}^1(\mathbb{C}_p) \setminus Z_n = \coprod_{i=1}^{s} B(x_i, \frac{1}{n})$

Let $\mathcal{O}(k, \mathcal{L})(\Omega)_n$ be the L-vs of functions $: Z_n \to L$ which are of the form:

$$\sum_{j=0}^{\infty} b_j z^j + \sum_{i=1}^{s} \sum_{j=1}^{\infty} \frac{b_{i,j}}{(z-x_i)^j} + \sum_{i=1}^{s} \sum_{j=0}^{k-2} c_{i,j} z^j \log_{\mathcal{L}}(z - x_i).$$

$$|b_i| n^{-i} \xrightarrow[i \to \infty]{} 0, \quad |b_{i,j}| n^i \xrightarrow[i \to \infty]{} 0$$

It's again a Banach. Let:

$\mathcal{O}(k, \mathcal{L})(\Omega) := \varprojlim_n \mathcal{O}(k, \mathcal{L})(\Omega)_n$. It's a Fréchet space.

G-action defined by:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} f \right)(z) = |ad-bc|^{-\frac{k-2}{2}} \frac{(a-cz)^{k-2}}{(ad-bc)^{k-2}} f\left(\frac{dz-b}{a-cz} \right)$$

<u>Lemma VI.3</u> (Breuil)

One has an exact sq of G-repr (strict):

$$0 \to \mathrm{Sym}^{k-2}(L^2) \to \mathcal{O}(k, \mathcal{L})(\Omega) \xrightarrow{\left(\frac{d}{dz}\right)^{\circ(k-1)}} \mathcal{O}(k)(\Omega) \to 0.$$

Exactness on the left and the middle is obvious, as are compatibility with the G-action and continuity of $\left(\frac{d}{dz}\right)^{\circ(k-1)}$. For the surjectivity, take $f \in \mathcal{O}(k)(\Omega)_{n+1}$ and integrate $k-1$ times, using $\log_{\mathcal{L}}$. Integration will create denominators, but the sum will converge in $\mathcal{O}(k, \mathcal{L})(\Omega)_n$. Then a Mittag-Leffler type argument gives the surjectivity on $\varprojlim$. Then apply the open mapping thm.

Th VI.4 (Breuil)

One has a G-eq topological iso:
$$\Pi(k,\mathcal{L}) \simeq \mathcal{O}(k+1,\mathcal{L})(\Omega)^*,$$
inserting in a comm. diag.:
$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}(k+1)(\Omega)^* & \to & \mathcal{O}(k+1,\mathcal{L})(\Omega)^* & \to & \underline{\mathrm{Sym}}^{k-1} & \to & 0 \\ & & \wr\Vert & & \wr\Vert & & \Vert & & \\ 0 & \to & \Pi(k) & \to & \Pi(k,\mathcal{L}) & \to & \underline{\mathrm{Sym}}^{k-1} & \to & 0 \end{array}$$
where: the top exact sequence is dual of the one in Lemma VI.3, the bottom one is the one defining $\Pi(k,\mathcal{L})$ (see before), and the left iso is Morita duality (Prop VI.2).

Rk: (a) ~~Let~~ Inside $\Pi(k)^*, \Pi(k,\mathcal{L})^*$ we have the subspaces $\widehat{\Pi(k)}^*, \widehat{\Pi(k,\mathcal{L})}^*$.
How to describe them geometrically? Choose U affinoid in Ω, s.t. $\bigcup_{g\in G} g.U = \Omega$.
then
$$\widehat{\Pi(k)}^* \simeq \{ f\in \mathcal{O}(k+1)(\Omega) \ , \ \|g.f|_U\|_\infty \leq 1 \ \forall g\in G\}\left[\tfrac{1}{p}\right] =: \mathcal{O}(k+1)(\Omega)^b$$
$$\widehat{\Pi(k,\mathcal{L})}^* \simeq \{ f\in \mathcal{O}(k+1,\mathcal{L})(\Omega), \ \|g.f|_U\|_\infty \leq 1 \ \forall g\in G\}\left[\tfrac{1}{p}\right] =: \mathcal{O}(k+1,\mathcal{L})(\Omega)^b.$$
Alternatively, here is another description of $\widehat{\Pi(k)}^*$. Deligne constructed a formal model $\widetilde{\Omega}$ of Ω with G action, and Grosse-Klonne a vector bundle G-equivariant $\widetilde{\mathcal{O}}(k)$ on $\widetilde{\Omega}$ $\forall k\in\mathbb{Z}$, s.t. $\widetilde{\mathcal{O}}(k)(\widetilde{\Omega})$ is a G-inv lattice in $\mathcal{O}(k)(\Omega)$. Then
$$\widehat{\Pi(k)}^* = \widetilde{\mathcal{O}}(k+1)(\widetilde{\Omega})\left[\tfrac{1}{p}\right].$$

(b) This description of $\widehat{\Pi(k)}^*$ is useful to show that $\Pi(k) = D(k)/\underline{\mathrm{Sym}}^{k-1}$ is not admissible, when $k>1$. One has to show that $\widetilde{\mathcal{O}(k+1)}(\widetilde{\Omega})$ is not of finite type over $\mathcal{O}_L[\![GL_2(\mathbb{Z}_p)]\!]$, and it suffices to show that $\widetilde{\mathcal{O}(k+1)}(\widetilde{\Omega})\otimes_{\mathcal{O}_L} k_L$ is not of finite type over $k_L[\![GL_2(\mathbb{Z}_p)]\!]$. To do so, one studies the special fiber of $\widetilde{\Omega}$, and show that $\widetilde{\mathcal{O}(k+1)}(\widetilde{\Omega})\otimes_{\mathcal{O}_L} k_L$ ~~contains~~ has a subquotient $\simeq \mathrm{ind}_{\mathbb{Q}_p^* GL_2(\mathbb{Z}_p)}^{G} \underline{\mathrm{Sym}}^r k_L^2$, $0\leq r\leq p-1$.
This rep is not of finite type over $k_L[\![GL_2(\mathbb{Z}_p)]\!]$.

(c) However with this geometric definition, without (φ,Γ)-modules, it seems very difficult to show that: $\widehat{\Pi(k,\mathcal{L})} \neq 0$; $\Pi(k,\mathcal{L}) \neq \Pi(k,\mathcal{L}')$ if $\mathcal{L}\neq\mathcal{L}'$.

Let $\Lambda(k) = \{ \mathcal{L}, \exists f$ primitive modular form of wt $k+1$ level pN, $N \wedge p = 1$ s.t. $\mathcal{L} = \mathcal{L}(\rho_{f,p}) \}$

For $\mathcal{L} \in \Lambda(k)$, Breuil gives a very nice global argument to show the two above points. The uniformization of Shimura curves by Ω and the Jacquet-Langlands correspondence (see next lecture) give the existence of $F \in \mathcal{O}(k+1)(\Omega)$, $\neq 0$, invariant par Γ from some discrete cocompact subgroup of G. (modulo center).

Let K be a c.o.s. of G stabilizing the affinoid U as above. Can assume that K preserves the unit ball of $\mathcal{O}(k+1)(U)$ (just replace the norm by an equivalent one, using the compacity of K). As Γ cocompact mod center, K open, the set $K\mathbb{Q}_p^\times \backslash G / \Gamma$ is finite. Choose representatives $g_1, \dots, g_r$ of the orbits, and $\lambda \in L$ s.t. $\|\lambda g_i \cdot F_{|U}\| \leq 1$ $\forall i = 1..s$. Then because K stabilizes the unit ball of $\mathcal{O}(k+1)(U)$ and because F is invariant by Γ, $\|g \cdot F_{|U}\| \leq 1$ $\forall g \in G$. Hence $F \in \mathcal{O}(k+1)(\Omega)^b$. In particular, $\widehat{\Pi(k)} \neq 0$!

Better, Breuil shows that actually $F \in \mathcal{O}(k+1, \mathcal{L})(\Omega)^b \underset{\uparrow}{\subsetneq} \mathcal{O}(k+1, \mathcal{L})(\Omega)^b$ iff $\mathcal{L} = \mathcal{L}(f)$. This proves the first point, and also the second (the multiplicity one).
same argument