LECT. VII: p-ADIC LLC & GEOMETRY: THE DE RHAM (NON SS) CASE

Let V be a 2-dim'l abs irr rep of $\mathcal{G}_{\mathbb{Q}_p}$, de Rham. So far, we understand very well $\Pi(V)$ and $\Pi(V)^{an}$ when V is semi-stable (and even: when V becomes potentially semi-stable over an abelian ext of $\mathbb{Q}_p$), because then V is trianguline, so these reps can be explicitly described, see lect IV, as spaces of functions on $\mathbb{Q}_p$. In the semi-stable non crystalline case we also got a geometric description of (the essential part) of $\Pi(V)^{an}$ in lect VI. Today, we'll see that although there is no simple description of $\Pi(V)^{an}$ when V is de Rham non semi-stable, we still have a geometric interpretation of this locally analytic representation.

Reformulation of the problem: let π be any (smooth) supercuspidal rep of G, $M(\pi)$ the $(\varphi, \mathcal{G}_{\mathbb{Q}_p})$-module associated to it by LLC (+Fontaine). Goal: find a geom description of $\Pi(V)^{an}$, when $V \in \mathcal{V}(\pi) := \{V \text{ de Rham}, \text{HT wts} = 0, 1, D_{pst}(V) \simeq M(\pi)\}$.

$\Leftrightarrow$ find a geometric description of Π^{an}, for any unitary admissible Banach rep of G, abs. irred & non ordinary, containing the smooth rep π.

Rk: (a) Of course, these two problems are the same, because of the p-adic LLC and its compatibility with classical LLC.

(b) We restrict to HT wts 0, 1 only for simplicity. From now on, all reps will have HT wts = 0, 1. Also assume $\det V = \chi$ (i.e. $\delta = 1$).

If V is semi-stable, it's not difficult to see that $\Pi(V)^{an}$ is an extension of a rep depending only on π by π (in the crystalline case this statement is empty anyway, as there's only one admissible filtration). In other words, the rep $\Pi(V)^{an}/\Pi(V)^{alg}$ is independent of the choice of a particular $V \in \mathcal{V}(\pi)$. I'll first explain briefly why this is actually true for all de Rham reps:

(Almost) Recall from lect II that if $V \in \cancel{\mathcal{V}(\pi)}$ is de Rham

$$D_{rig}(V) = \left\{ z \in \left(\mathcal{R}_K[\tfrac{1}{t}, \log T] \otimes D_{pst}(V)\right)^{\mathcal{G}_K}{}^{\mathrm{Gal}(K^{cyc}/\mathbb{Q}_p^{cyc}),\, N=0},\ \varphi^{-n}(z) \in \mathrm{Fil}^0(L_n((t)) \otimes D_{dR}(V))\ \forall n \gg 0 \right\}$$

where K is any extension s.t. $V_{|\mathcal{G}_K}$ is semi-stable.

In particular, because the HT wts of V are $0, 1$, we see that the (φ, Γ)-module over $\mathcal{R}$:

$$N_{rig}(V) = \left(\mathcal{R}_K[\log(T)] \otimes D_{pst}(V)\right)^{N=0,\ \mathrm{Gal}(K^{cyc}/\mathbb{Q}_p^{cyc})}$$

is such that $\quad t N_{rig}(V) \subseteq D_{rig}(V)$.

Let: $tN_{rig}(V) \boxtimes \mathbb{P}^1(\mathbb{Q}_p) = \{ z \in D_{rig}(V) \boxtimes \mathbb{P}^1(\mathbb{Q}_p),\ \mathrm{Res}_{\mathbb{Z}_p}(z), \mathrm{Res}_{\mathbb{Z}_p}(wz) \in tN_{rig}(V)\}$

Note that this is really a definition, not a proposition, as $tN_{rig}(V)$ is not étale.

<u>Th</u> <u>VII</u>.1 (Colmez)

The subspace $tN_{rig}(V) \boxtimes \mathbb{P}^1(\mathbb{Q}_p)$ of $D_{rig}(V) \boxtimes \mathbb{P}^1(\mathbb{Q}_p)$ is stable by the action of G and this representation of G is independent of the choice of $V \in \mathcal{V}(\pi)$. It decomposes as an extension:

$$0 \to \Pi(\pi, 0)^* \to tN_{rig} \boxtimes \mathbb{P}^1(\mathbb{Q}_p) \to \Pi(\pi, 2) \to 0,$$

$\Pi(\pi, 0)$ & $\Pi(\pi, 2)$ being two loc. an repr of G. Any choice of an isomorphism $D_{pst}(V) \simeq M(\pi)$ for $V \in \mathcal{V}(\pi)$ gives an isom $\quad \Pi(\pi, 0) \simeq \Pi(V)^{an}/\Pi(V)^{alg}$.

<u>Rk</u>: (a) The exact sequence of the thm is of course in the same spirit that the exact sequence of Th. <u>IV</u>.6.

(b) If $M(\pi)$ is irreducible (which will be the case of interest), the rep $\Pi(V)^{an}/\Pi(V)^{alg}$ is canonically isomorphic to $\Pi(\pi, 0)$, up to scalar.

In summary, for any $V \in \mathcal{V}(\pi)$, $\Pi(V)^{an}$ is an extension:

$$0 \to \pi \to \Pi(V)^{an} \to \Pi(\pi, 0) \to 0.$$

First goal: give a geometric description of $\Pi(\pi, 0)$.

From now on, assume π <u>supercuspidal</u> and let $\rho = JL(\pi) \circlearrowleft D^\times$ $(\rho \neq 1)$.

Recall the Drinfeld tower:

$$\begin{array}{c} \mathcal{M}_n \\ \downarrow \\ \mathcal{M}_0 \simeq \hat{\Omega} \times \mathbb{Z} \\ \downarrow \\ \hat{\Omega} \end{array}$$

with actions commuting of G and $D^\times$.

$\mathcal{M}_n/(p)^{\mathbb{Z}}$ descends to a rigid analytic space over $\mathbb{Q}_p$, called Σ_n.

Σ_n is a Stein space, and thus $\mathcal{O}(\Sigma_n)^*$ is a space of compact type.

<u>Prop</u> <u>VII</u>. 2: The G-rep $\mathcal{O}(\Sigma_n)^*$ is a locally analytic rep of G.

($\Leftrightarrow \mathcal{O}(\Sigma_n)$ is a separately C^0 module over $\mathcal{O}(G)$).

Let $\mathcal{O}(\Sigma_n)^\rho = \mathrm{Hom}_{D^\times}(\rho, \mathcal{O}(\Sigma_n))$. Chose any n s.t. ρ is trivial on $1+p^n\mathcal{O}_D$.

Th VII.3 : $\exists$ (unique up to scalar) isomorphism of locally an G-reps:

$$\Pi(\pi,0) \simeq (\mathcal{O}(\Sigma_n)^\rho)^*.$$

Rk: You can see this statement as giving a geometric realization of $\Pi(\pi,0)$, but also the other way, as a complete description of the space of functions on Σ_n as a $G\times D^\times$-rep in terms of p-adic LLC & (classical) JLC.

Cor VII.4 : The $D(G)$-module rep $\mathcal{O}(\Sigma_n)^\rho$ is coadmissible.

The proof of Th VII.3 has two steps:

(A) Construct a G-eq $C^\circ_{\neq 0}$ morphism $\Phi : \Pi(\pi,0)^* \to \mathcal{O}(\Sigma_n)^\rho$.

(B) Show that such a morphism Φ is necessarily an isomorphism.

Part (A) uses global arguments:

Let B be a quaternion algebra over $\mathbb{Q}$, ramified at p $(B\otimes_{\mathbb{Q}}\mathbb{Q}_p \simeq D)$ split at ∞.

Associated to B is a family of alg var $/\mathbb{Q}$ of dim 1 $(Sh_K)_{K\subset B^\times(\mathbb{A}_f)}$ (Shimura curve).

$$\text{s.t.}\quad Sh_K(\mathbb{C}) \simeq B^\times(\mathbb{Q})\backslash\big((\mathbb{C}\setminus\mathbb{R})\times B^\times(\mathbb{A}_f)/K\big)$$

(analogue of the modular curve, with B instead of GL_2).

By def, a Shimura curve is uniformized by the archimedean half-plane $\mathbb{C}\setminus\mathbb{R}$. But the hypothesis B ramified at p allows also to uniformize (the rigid analytic variety attached to) Sh_K by Drinfeld's half-plane and its coverings: one has an iso:

$$\bar{B}^*(\mathbb{Q})\backslash\big(\mathcal{M}_n\times B^*(\mathbb{A}_f^p)/K^p\big) \simeq \big(Sh_K\otimes_{\mathbb{Q}_p}\breve{\mathbb{Q}}_p\big)^{an}$$

if $K=(1+p^n\mathcal{O}_D).K^p$ and $\bar{B}$ is the quaternion algebra $/\mathbb{Q}$ locally isomorphic to B at all places $\neq p,\infty$, and split at p and ramified at ∞. It acts diagonally on $\mathcal{M}_n\times B^*(\mathbb{A}_f^p)$: on $\mathcal{M}_n$ thru $\bar{B}^*(\mathbb{Q})\hookrightarrow\bar{B}^*(\mathbb{Q}_p)\simeq G$ and on $B^*(\mathbb{A}_f^p)\simeq\bar{B}^*(\mathbb{A}_f^p)$.

(This is the Cerednik-Drinfeld uniformization, generalized by Rapoport-Zink to many Rapoport-Zink spaces).

$$\Rightarrow \quad \Omega^1(Sh_K)^\rho \simeq \mathrm{Hom}_G^{cont}\Big(\big(\Omega^1(\Sigma_n)^\rho\big)^*, LA(\underbrace{X(K^p)}_{\substack{\| \\ \bar{B}^\times(\mathbb{Q})\backslash \bar{B}^\times(\mathbb{A}_f)/K^p \\ \text{profinite set}}})\Big)$$

Choose a modular form of wt 2 f s.t. $\pi(f)_p \simeq \pi$.

Let $\mathfrak{p}$ be the ideal of the Hecke algebra away from p corresponding to f.

$\Omega^1(Sh_K)^\rho[\mathfrak{p}]$ is one-dimensional, generated by the quaternionic modular form attached to f by the (global) JLC (viewed as a differential form on Sh_K).

Now, $LA(X(K^p))$ is the space of locally analytic vectors of the completed cohomology of the Shimura variety associated to $\bar{B}$ (which is of dim 0). As $\bar{B}$ is split at p, you can in fact in fact copy Emerton's proof of the local-global compatibility to get:

$$LA(X(K^p))[\mathfrak{p}] = \Pi(\rho_{f,p})^{an\,\oplus r} \qquad \begin{array}{l} (\text{ref. V}) \\ \text{some } r>0 \\ (K^p \text{ small enough}). \end{array}$$

This gives you a G-eq C^0 morphism $\Phi^\vee : \mathcal{O}(\Sigma_n)^\rho \to \Pi(\rho_{f,p})^{an} / \Pi(\rho_{f,p})^{alg}$
(little extra work)
$$\Pi(\rho_{f,p})^{an} / \Pi(\rho_{f,p})^{alg} \simeq \Pi(\pi,0).$$

Then dualize (everything is reflexive).

Part (B) is purely local. I'll sketch the proof of:

$$\Phi \text{ has dense image.} \qquad (*)$$

Not easy to understand $\mathcal{O}(\Sigma_n)^\rho$ as a $D(G)$-module. But it's obviously a coadmissible module over the Fréchet-Stein algebra $\mathcal{O}(\Omega)$. Why not trying to put a $\mathcal{O}(\Omega)$-module structure on $\Pi(\pi,0)^*$, which is easier to handle?

Prop VII.5: $\exists!$ automorphism ∂ of the top L-vs $\Pi(\pi,0)^*$ s.t.

$$a^+ - 1 = u^+ \circ \partial \qquad \text{where } a^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ u^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}_2.$$

Rk: The operator ∂ comes from a connection called ∂ by Berger on the (φ,Γ)-module $N_{rig}(V)$, whence the name.

Prop VII.6: $\exists!$ $\mathcal{O}(\Omega)$-module structure on $\Pi(\pi,0)^*$ compatible with its L-vs structure & st. $z \in \mathcal{O}(\Omega)$ acts as ∂.

Rk: Prop VII.5 and the def of ∂ were of course motivated by the analogous formulas for "the multiplication by z" operator on $\mathcal{O}(\Sigma_n)$.

To prove Prop VII.6, one uses Morita duality (Prop VI.4):

let $f \in \mathcal{O}(\Omega)$, write $f(z) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{1}{z-x} \mu(x)$, $\mu \in \mathcal{D}^0(\mathbb{P}^1(\mathbb{Q}_p), L)$.

let $v \in \Pi(\pi_{1,0})^*$, and then show that:

$$f \cdot v = \int_{\mathbb{P}^1(\mathbb{Q}_p)} (\partial - x)^{-1} \cdot v \, \mu(x) \quad \text{makes sense.}$$

Back to the sketch of pf of $(*)$:

Set $W = \overline{\mathrm{Im}(\Phi)} \subseteq \mathcal{O}(\Sigma_n)^\rho$.

$\mathcal{O}(\Sigma_n)^\rho$ is coadmissible over $\mathcal{O}(\Omega)$, and W is closed in $\mathcal{O}(\Sigma_n)^\rho$, so it's also coadmissible over $\mathcal{O}(\Omega)$.

As Ω is Stein, the exact sequence of G-reps: $0 \to W \to \mathcal{O}(\Sigma_n)^\rho \to \mathcal{O}(\Sigma_n)^\rho / W \to 0$ corresponds to an exact sequence of G-eq coherent sheaves:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{G}/\mathcal{F} \to 0$$

which are actually all vector bundles.

Kohlhaase: $\exists$ exact functor inducing an equivalence of categories:

$$\left\{ \begin{array}{l} \text{"certain" } G\text{-eq} \\ \text{vector bundles on } \Omega \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{"certain" } D^\times\text{-eq} \\ \text{vb on } \mathbb{P}^1 \end{array} \right\}$$

s.t. $\mathcal{G} \leftrightarrow \underbrace{\rho^* \otimes \mathcal{O}_{\mathbb{P}^1}}_{\text{irreducible as } D^\times\text{-eq vb !}}$.

$\Rightarrow \quad W = \mathcal{O}(\Sigma_n)^\rho$, i.e. $(*)$ is true. $\square$.

Thm VII.2 gives a geometric description of $\Pi(\pi_{1,0})$; what about $\Pi(V)^{an}$?

Σ_n is Stein, so it has no coherent coh. in positive degrees. Hence we have an exact sequence of $G \times D^\times$-reps:

$$0 \to L \to \mathcal{O}(\Sigma_n) \xrightarrow{d} \Omega^1(\Sigma_n) \to H^1_{dR}(\Sigma_n) \to 0.$$

Take the ρ-isotypic part. As $\rho \neq 1$, get:

$$0 \to \mathcal{O}(\Sigma_n)^\rho \xrightarrow{d} \Omega^1(\Sigma_n)^\rho \to H^1_{dR}(\Sigma_n)^\rho \to 0.$$

As a G-rep, $\Omega^1(\Sigma_n)$ is simpler: $f \mapsto f\,dz$ identifies $\mathcal{O}(\Sigma_n)$ and $\Omega^1(\Sigma_n)$ as top. v.s., and the action of G is given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (f\,dz) = (ad-bc)(a-cz)^{-2}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot f\right) dz .$$

Can now play the same game on the (φ,Γ)-modules side: define the G-rep $\Pi(\pi,2)^*$ by letting G act on the t.v.s $\Pi(\pi,0)^*$ by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} * v = (ad-bc)(a-c\partial)^{-2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot v$$

The morphism $d: \mathcal{O}(\Sigma_n) \to \Omega^1(\Sigma_n)$ is G-eq and corresponds to the action of $-u^+ \in \mathfrak{gl}_2$. Similarly, one checks that $-u^+ : \Pi(\pi,0)^* \to \Pi(\pi,2)^*$ is G-eq, whence an exact sequence of G-reps:

$$0 \to \Pi(\pi,0)^* \xrightarrow{-u^+} \Pi(\pi,2)^* \to \left(\Pi(\pi,2)^{u^+=0}\right)^* \to 0$$

which gets identified with:

$$0 \to \mathcal{O}(\Sigma_n)^\rho \to \Omega^1(\Sigma_n)^\rho \to H^1_{dR}(\Sigma_n)^\rho \to 0$$

through the isomorphism Φ of Th. VII.2.

Th VII.7. (a) There is a unique up to scalar isomorphism of G-reps:

$$H^1_{dR}(\Sigma_n)^\rho \simeq \pi^* \otimes M_{dR}(\pi)^*$$

$\left(M_{dR}(\pi) := (M(\pi)\otimes \overline{\mathbb{Q}}_p)^{\mathcal{G}_{\mathbb{Q}_p}}\right)$.

(b) Let $\mathcal{L} \in \mathrm{Proj}(M_{dR}(\pi))$. It gives an admissible filtration on $M(\pi)$, hence by Colmez-Fontaine's thm I.4, a rep $V_{\mathcal{L}} \in \mathcal{V}(\pi)$. Then the inverse image of $\mathcal{L}^\perp \otimes \pi^* \subseteq H^1_{dR}(\Sigma_n)^\rho$ in $\Omega^1(\Sigma)^\rho$ is isomorphic to $(\Pi(V_{\mathcal{L}})^{an})^*$.

In other words, we have a diagram:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}(\Sigma_n)^\rho & \to & \Omega^1(\Sigma_n)^\rho & \to & H^1_{dR}(\Sigma_n)^\rho & \to & 0 \\ & & \parallel & & \cup\!\text{l} & & \cup\!\text{l} & & \\ 0 & \to & \mathcal{O}(\Sigma_n)^\rho & \to & (\Pi(V_{\mathcal{L}})^{an})^* & \to & \mathcal{L}^\perp \otimes \pi^* & \to & 0 \end{array}$$

Rk: (a) The subspace $\mathcal{L}^\perp \otimes \pi^*$ of $H^1_{dR}(\Sigma_n)^\rho$ is well defined, as the isom of (a) is unique up to scalar.

(b) If you dualize the last line of the diagram, you recover the exact sequence: $0 \to \pi \to \Pi(V_{\mathcal{L}})^{an} \to \Pi(\pi,0) \to 0$.

Here are a few questions suggested by the last theorem.

Q1. Σ_n is Stein, so we can apply a similar argument as in lect VI to obtain an exact sequence of $D^\times G$-repr:

$$0 \to \mathcal{O}(\Sigma_n) \to \varinjlim_{\substack{Z \text{ fin union} \\ \text{of adm aff.}}} \mathcal{O}(\Sigma_n \setminus Z) \to \Omega^1(\Sigma_n)^* \to 0.$$

Let (U_i) be a Stein cover of Ω, and let $\pi_n : \Sigma_n \to \Omega$ the canonical map. Have an exact sequence of G-repr:

$$0 \to \mathcal{O}(\Sigma_n)^\rho \to \varinjlim \mathcal{O}(\Sigma_n \setminus \pi_n^{-1}(U_i))^\rho \to \Omega^1(\Sigma_n)^{\rho *} \to 0$$

Is this exact sequence identified with (Thm VII.1):

$$0 \to \Pi(\pi, 0)^* \to \hat{\Pi}_{\mathrm{Nrig}} \boxtimes \mathbb{P}^1(\mathbb{Q}_p) \to \Pi(\pi, 2) \to 0 \quad ?$$

Can even define a G-eq sheaf on $\mathbb{P}^1(\mathbb{Q}_p)$ = boundary of Ω by a similar process, and ask if it identifies with $U \mapsto \hat{\Pi}_{\mathrm{Nrig}} \boxtimes U$.

Q2: The inverse limit of the spaces Σ_n makes sense as a perfectoid space Σ_∞ which has a moduli interpretation and a nice description, due to Fargues & Scholze-Weinstein. $\mathcal{O}(\Sigma_\infty)$ is a completion of $\varinjlim \mathcal{O}(\Sigma_n)$: take (U_i) a Stein cover of Ω. For all n, $\mathcal{O}(\pi_n^{-1}(U_i))$ is a Banach and as the transition maps in the tower are finite étale, the induced maps $\mathcal{O}(\pi_n^{-1}(U_i)) \to \mathcal{O}(\pi_{n+1}^{-1}(U_i))$ are isometries. Hence we get a well defined topology on the inductive limit. Then take $\varprojlim_i$. Complete wrt this topology.

What does this big rep of $G \times D^\times$ look like? What I explained before gives a complete description of the $D^\times$-smooth part of $\mathcal{O}(\Sigma_\infty)$. But we should see also plenty of loc an repr of $D^\times$!

Q3: ~~Both~~ In the semi-stable non crystalline case and in the potentially crystalline (non crystabelline) case, we have a geometric description of the locally analytic vectors. But the recipes are quite different!

You cannot just copy Th. $\underline{\underline{\text{VII}}}$.7 when $\rho = 1$, because

$$\forall n \geq 0 \quad H^1_{dR}(\Sigma_n)^{D^\times} = H^1_{dR}(\Omega) \simeq St^*$$

and not to $St^* \otimes Sp^*$.

(See remark (c) after Th $\underline{\text{VI}}$.4: some phenomenon).

Still, can one find a uniform treatment of both cases?

In this direction, all I have to offer is:

Guess: it should be true that $\forall$ any irred ρ of D^*, $\forall V \in \mathcal{V}^g(\pi)$,

$$\mathrm{Hom}_{\underbrace{D^b(D(G))}_{}}\left((\Pi(V)^{an})^*, \; R\Gamma_{rig}(\Sigma_n)^\rho[1] \right) = D_{pst}(V).$$

as $(\varphi, N, \mathcal{G}_{\mathbb{Q}_p})$-modules.

bdd derived category of the abelian cat. of coadmissible $D(G)$-mod.

Here $R\Gamma_{rig}(\Sigma_n)$ should be the complex of log-rigid cohomology of Σ_n. (the difficulty is to define it as an object of $D^b(D(G))$).

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(a generalization of crystalline coh. to non proper non smooth formal schemes)

If $\rho = 1$, this guess (slightly modified) is a theorem of Schraen.

If $\rho \neq 1$, it should be a consequence of my joint work with Dospinescu, once the conceptual difficulty of defining $R\Gamma_{rig}(\Sigma_n)$ will be solved...